ZETA FUNCTIONS OF FINITELY GENERATED VIRTUALLY NILPOTENT GROUPS

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ABSTRACT. We prove that the subgroup and the normal zeta functions of virtually nilpotent groups can be written as a finite sum of Euler products of cone integrals and we deduce from this that they have rational abscissas of convergence and meromorphic continuation to the left. We give a method to obtain the cone conditions for these cone integrals and as an application we compute the zeta functions of all finitely generated torsion free virtually nilpotent groups of Hirsch length 3.

INTRODUCTION

For a finitely generated group G, let $a_n^{\leq}(G)$ and $a_n^{\lhd}(G)$ denote respectively the number of subgroups of G of index n and the number of normal subgroups of G of index n. The zeta function of G is defined as the Dirichlet series:

$$\zeta_G^{\leq}(s) = \sum_{n=1}^{\infty} \frac{a_n^{\leq}(G)}{n^s} = \sum_{A \leq_f G} [G : A]^{-s},$$

where $\leq_f$ means "subgroup of finite index". The local zeta function of G at a prime p is defined as:

$$\zeta_{G,p}^{\leq}(s) = \sum_{k=0}^{\infty} \frac{a_{p^k}^{\leq}(G)}{p^{ks}} = \sum_{A \leq_p G} [G : A]^{-s},$$

where $\leq_p$ means "subgroup of index a power of p". The normal zeta function of G and the local normal zeta function of G at p, denoted by $\zeta_G^{\lhd}(s)$ and $\zeta_{G,p}^{\lhd}(s)$ respectively, are defined similarly using $a_n^{\lhd}(G)$ instead of $a_n^{\leq}(G)$. To avoid repetitions we shall use the symbol $*$ whenever we refer to both $\leq$ and $\lhd$.

These series were introduced in [GSS] as a tool to study the arithmetic properties of the sequences $a_n^{\leq}(G)$ and $a_n^{\lhd}(G)$, their growth and their asymptotic behavior. The series $\zeta_G^{*}(s)$ defines an analytic function

Date: March 4, 2013.
1991 *Mathematics Subject Classification.* 20E07, 11M41.

in the (non-empty) region $\Re(s) > \alpha^{\leq}(G)$ when $\alpha^{\leq}(G)$, the abscissa of convergence of $\zeta^*_G(s)$, is finite (or even $-\infty$) or equivalently when G has polynomial subgroup growth. A finitely generated residually finite group G has polynomial subgroup growth if and only if it contains a finite index subgroup which is soluble of finite rank [LMS]. The class of finitely generated virtually nilpotent groups (containing a nilpotent normal subgroup of finite index) have this property and in particular the smaller class of τ-groups (finitely generated torsion free nilpotent groups).

The following results for τ-groups were obtained in [GSS] and [dSG]:

I. $\zeta^*_G(s)$ has rational abscissa of convergence $\alpha^*(G) \leq h(G)$, where $h(G)$ is the Hirsch length of G, and this number depends only on the $\mathbb{Q}$-Mal'cev completion of G.

II. $\zeta^*_G(s)$ has meromorphic continuation to the left, that is, there exists $\delta > 0$ such that $\zeta^*_G(s)$ can be extended meromorphically to the region $\{s \in \mathbb{C} : \Re(s) > \alpha^*(G) - \delta\}$.

III. $\zeta^*_G(s)$ has Euler product decomposition: $\zeta^*_G(s) = \prod_{p \text{ prime}} \zeta^*_{G,p}(s)$, and the local factors $\zeta^*_{G,p}(s)$ are rational functions in p^{-s}; more precisely, for each prime p, there exist polynomials P_p and Q_p of bounded degree with coefficients in $\mathbb{Q}$ such that $\zeta^*_{G,p}(s) = \frac{P_p(p^{-s})}{Q_p(p^{-s})}$.

The goal of this paper is to extend these results to the class of (finitely generated) virtually nilpotent groups. Notice that this family coincides with the family of virtually τ-groups (containing a normal subgroup of finite index which is a τ-group).

The zeta functions of finite extensions of the free abelian group $\mathbb{Z}^d$, a proper subclass of virtually nilpotent groups, were studied in [dSMS]. They first observed that if N is a normal subgroup of G of finite index, then one has that

$$\zeta^*_G(s) = \sum_{N \leq H \leq G} \left(\sum_{\substack{A *_f G \\ AN = H}} [G:A]^{-s} \right) = \sum_{N \leq H \leq G} [G:H]^{-s} \left(\sum_{\substack{A *_f G \\ AN = H}} [H:A]^{-s} \right).$$

Then they proved that if N is finitely generated torsion free abelian, then for each $N \leq H \leq G$ there exists an Euler product decomposition

[dSMS, Propositions 2.2 and 2.5]:

$$\sum_{\substack{A *_f G \\ AN=H}} [H:A]^{-s} = \prod_{p \ prime} \left(\sum_{\substack{A * G, A \leq_p H \\ AN=H}} [H:A]^{-s} \right).$$

C. Voll observed that in the proof only the fact that the profinite completion of N is a product over all primes p of its Sylow pro-p subgroups was used and this is also the case if we just assume that N is a τ-group.

Therefore, as in [dSMS], to study the analytic properties of $\zeta^*_G(s)$, it is enough to fix $N \leq H \leq G$ and consider the following zeta function associated to the pair (H, N):

$$\zeta^*_{H,N}(s) := \sum_{\substack{A *_f G \\ AN=H}} [H:A]^{-s} = \prod_{p \ prime} \zeta^*_{H,N,p}(s),$$

where

$$\zeta^*_{H,N,p}(s) = \sum_{\substack{A * G, \ A \leq_p H \\ AN=H}} [H:A]^{-s}.$$

Several analytic properties of $\zeta^*_{H,N}(s)$ can be obtained by expressing it as an Euler product of cone integrals over $\mathbb{Q}$. Given a natural number m, a finite collection of polynomials

$$\mathcal{D} = (f_0, g_0; f_1, g_1, \dots, f_l, g_l)$$

with $f_0, g_0, f_1, g_1, \dots, f_l, g_l \in \mathbb{Q}[x_1, \dots, x_m]$ is called a cone integral data. For each prime p, one associates to $\mathcal{D}$ the closed subset of $\mathbb{Z}_p^m$

$$\mathcal{M}(\mathcal{D}, p) = \{\mathbf{x} \in \mathbb{Z}_p^m : v_p(f_i(\mathbf{x})) \leq v_p(g_i(\mathbf{x})) \text{ for } i = 1, \dots, l\},$$

where v_p is the p-adic valuation on $\mathbb{Z}_p$. Then the p-adic integral

$$Z_{\mathcal{D}}(s, p) = \int_{\mathcal{M}(\mathcal{D},p)} |f_0(\mathbf{x})|_p^s |g_0(\mathbf{x})|_p d\mu(\mathbf{x}),$$

where μ is the normalized Haar Measure on $\mathbb{Z}_p^m$ and s is a complex number, is called a *cone integral* defined over $\mathbb{Q}$. By a result of Denef [De], each $Z_{\mathcal{D}}(s, p)$ is a rational function in p^{-s} and therefore it can be written as a power series

$$Z_{\mathcal{D}}(s, p) = \sum_{i=0}^{\infty} a_{p,i}(\mathcal{D}) p^{-is}$$

with non-negative coefficients. A function $Z(s)$ is said to be an Euler product of cone integrals over $\mathbb{Q}$ with cone integral data $\mathcal{D}$ if

$$Z(s) = \prod_{\substack{p \ prime \\ a_{p,0}(\mathcal{D})\neq 0}} (a_{p,0}^{-1}(\mathcal{D}) \cdot Z_{\mathcal{D}}(s,p)),$$

and in this case one writes $Z(s) = Z_{\mathcal{D}}(s)$.

In [dSG] it is proved that such a function $Z(s)$ is expressible as a Dirichlet series $\sum_{n=1}^{\infty} a_n n^{-s}$ with nonnegative coefficients and that if for almost all primes p the function $Z_{\mathcal{D}}(s,p)$ is not the constant function, then:

I. The abscissa of convergence $\alpha_{\mathcal{D}}$ of $Z_{\mathcal{D}}(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ is a rational number.
II. $Z_{\mathcal{D}}(s)$ has meromorphic continuation to $\Re(s) > \alpha_{\mathcal{D}} - \delta$ for some $\delta > 0$.
III. The continued function is holomorphic on the line $\Re(s) = \alpha_{\mathcal{D}}$ except for a possible pole at $s = \alpha_{\mathcal{D}}$.
IV. The abscissa of convergence of each local factor is strictly to the left of $\alpha_{\mathcal{D}}$.

In the first half of the paper we prove the following results.

Theorem A. *Let G be a finitely generated virtually nilpotent group, N a normal subgroup of G of finite index which is a τ-group, and let $N \leq H \leq G$. Then there exists a cone integral data $\mathcal{D}^*$ such that $\zeta^*_{H,N}(s)$ is the Euler product of cone integrals over $\mathbb{Q}$:*

$$\zeta^*_{H,N}(s) = Z_{\mathcal{D}^*}(s - h(N) - [H:N] + 1).$$

Theorem B. *Let G be a finitely generated virtually nilpotent group and N a finite index normal subgroup of G which is a τ-group. Then*

(1) $\zeta^*_G(s)$ has rational abscissa of convergence $\alpha^*(G) \leq \alpha^*(N) + 1 \leq h(G) + 1$.
(2) There exists $\delta > 0$ such that $\zeta^*_G(s)$ has meromorphic continuation to the region $\Re(s) > \alpha^*(G) - \delta$, and the line $\Re(s) = \alpha^*(G)$ contains at most one pole of $\zeta^*_G(s)$ (at the point $s = \alpha^*(G)$).
(3) $\zeta^*_G(s) = \sum_{N \leq H \leq G} [G:H]^{-s} \zeta^*_{H,N}(s)$ and each $\zeta^*_{H,N}(s)$ has Euler product decomposition $\zeta^*_{H,N}(s) = \prod_{p \ prime} \zeta^*_{H,N,p}(s)$, where each $\zeta^*_{H,N,p}(s)$ is a rational function in p^{-s}.

In the course of proving Theorem A we deduce the following corollary.

Corollary 1. *The zeta function and the normal zeta function of a τ-group are Euler products of cone integrals over $\mathbb{Q}$.*

We observe that this corollary improves a result obtained in [dSG, Remark after Corollary 5.6] which says that for a τ-group G,

$$\zeta_G^*(s) = Z_{\mathcal{D}^*}(s - h(G))P(s),$$

where $Z_{\mathcal{D}^*}(s)$ is an Euler product of cone integrals over $\mathbb{Q}$ and $P(s) = \prod_{p \in S} P_p(p^{-s})$, where S is a finite set of primes and $P_p(x)$ is a rational function.

In the second half of the paper we introduce the $\mathbb{Q}$-Mal'cev completion of a pair (G, N), where G is a group and N a finite index normal subgroup of G which is a τ-group. It is a group K, uniquely determined by (G, N) up to isomorphism, containing both G as a subgroup and the $\mathbb{Q}$-Mal'cev completion $N^{\mathbb{Q}}$ of N as a normal subgroup of finite index in such a way that $N^{\mathbb{Q}}G = K$ and $N^{\mathbb{Q}} \cap G = N$. In general, if K is any group containing a finite index normal subgroup M which is the $\mathbb{Q}$-Mal'cev completion of some τ-group, then K is the $\mathbb{Q}$-Mal'cev completion of any pair $(G, G \cap M)$, where G is a subgroup of K such that $GM = K$ and $G \cap M$ is a τ-group with $\mathbb{Q}$-Mal'cev completion M.

We are able to state the following theorem, which is the analogous of Proposition 3 in [GSS].

Theorem C. *Let G be a finitely generated virtually nilpotent group and let N be a finite index normal subgroup of G which is a τ-group. Then the abscissa of convergence of $\zeta_{G,N}^{\leq}(s)$ depends only on the $\mathbb{Q}$-Mal'cev completion of (G, N). In particular, the abscissa of convergence of $\zeta_G^{\leq}(s)$ depends only on the $\mathbb{Q}$-Mal'cev completion of (G, N).*

The first application is the obtention of formulas for the abscissa of convergence of the zeta functions of virtually abelian groups.

The second application is the explicit computation of the zeta functions of all 3-dimensional Bieberbach groups, that is, torsion free virtually nilpotent groups of Hirsch length 3. These groups were classified in [DIKL]. In this paper we include only the results. The complete computations are in [S].

1. Local factors as cone integrals

From now on we fix a finitely generated virtually nilpotent group G, a normal subgroup N of index r which is a τ-group of Hirsch length $h \geq 1$ and a subgroup H with $N \leq H \leq G$ and $[H : N] = r_1$. We

consider the following two zeta functions:

$$\zeta^{\leq}_{H,N}(s) = \sum_{\substack{A \leq_f H \\ AN=H}} [H:A]^{-s} \text{ and } \zeta^{\lhd}_{H,N}(s) = \sum_{\substack{A \lhd_f G \\ AN=H}} [H:A]^{-s}.$$

Since the second zeta function is zero if H is not normal in G, then we shall always assume that H is normal in G whenever we are considering the case $* = \lhd$.

For each prime p, let $\mathcal{N}_p$ be the family of normal subgroups of G contained in N and which are of index a power of p in N, and let G_p be the inverse limit of the quotients G/A for $A \in \mathcal{N}_p$. Observe that if $B \leq_p N$ then the subgroup $\cap_{g \in G} B^g$ is in $\mathcal{N}_p$, and since N is a residually-(finite p-group) then the intersection of all the elements of $\mathcal{N}_p$ is $\{1\}$. Hence G can be viewed as a dense subgroup of G_p naturally and in this way we shall denote by N_p and H_p the closures of N and H in G_p respectively. The fact that the family of subgroups of N of index a power of p is cofinal with $\mathcal{N}_p$ implies that N_p is the pro-p completion of N.

Let $\mathcal{H}_p$ be the family of all the subgroups of G containing some element of $\mathcal{N}_p$ and let $\mathcal{O}_p$ be the family of the open subgroups of G_p. Observe that $\mathcal{H}_p$ is precisely the family of subgroups A of G for which $A \cap N \leq_p N$, and $\mathcal{O}_p$ is the family of the finite index subgroups of G_p. The latter is because for $B \leq_f G_p$ we have $B \cap N_p \leq_f N_p$ and therefore $B \cap N_p$ is open since N_p is a finitely generated pro-p group ([DdSMS, Theorem 1.17]), and moreover $B \cap N_p \leq_p N_p$. It is a general fact in profinite completions that the map $A \to \overline{A}$ gives a correspondence between $\mathcal{H}_p$ and $\mathcal{O}_p$ with inverse map given by $B \to B \cap G$, and a subgroup $A \in \mathcal{H}_p$ is normal if and only if $\overline{A}$ is normal. This correspondence is also index-preserving in the sense that $[G_p : \overline{A}] = [G : A]$ for all $A \in \mathcal{H}_p$.

Proposition 1.1. *Let $A \leq_f G$. Then the map $A_1 \to \overline{A_1}$ gives an index-preserving correspondence between the family $\mathcal{H}_p(A)$ of subgroups $A_1 \leq_p A$ for which $A_1(A \cap N) = A$ and the family $\mathcal{O}_p(\overline{A})$ of open subgroups B of $\overline{A}$ for which $B(\overline{A \cap N}) = \overline{A}$. Under this correspondence, normal subgroups of A correspond to normal subgroups in $\overline{A}$ and if A is normal in G then normal subgroups of G correspond to normal subgroups of G_p.*

Proof. First we shall prove that $[\overline{A} : \overline{A_1}] = [A : A_1]$ for all $A_1 \in \mathcal{H}_p(A)$. In fact, for $A_1 \in \mathcal{H}_p(A)$ we have $[A \cap N : A_1 \cap N] = [A : A_1]$ which is a power of p, and since N is a τ-group it is a well known fact that $[\overline{A \cap N} : \overline{A_1 \cap N}] = [A \cap N : A_1 \cap N]$. This implies that

$[A : A_1] = [\overline{A \cap N} : \overline{A_1 \cap N}]$. Now $A_1(\overline{A \cap N})$ is closed and contains A, and conversely $\overline{A}$ contains $A_1(\overline{A \cap N})$ clearly. Then $\overline{A} = A_1(\overline{A \cap N})$ and similarly $\overline{A_1} = A_1(\overline{A_1 \cap N})$ (*). Then $[\overline{A} : \overline{A_1}] = [A_1(\overline{A \cap N}) : A_1(\overline{A_1 \cap N})] = [\overline{A \cap N} : \overline{A \cap N} \cap A_1(\overline{A_1 \cap N})]$ and in order to show that $\overline{A \cap N} \cap A_1(\overline{A_1 \cap N}) = \overline{A_1 \cap N}$, it is enough to prove that $A_1 \cap (\overline{A \cap N}) \leq A_1 \cap N$. But $A_1 \cap (\overline{A \cap N}) \leq A_1 \cap N_p \leq G \cap N_p = N$ and therefore $A_1 \cap (\overline{A \cap N}) \leq A_1 \cap N$. We conclude that $[\overline{A} : \overline{A_1}] = [A : A_1]$ and the expressions for $\overline{A}$ and $\overline{A_1}$ in (*) imply that $\overline{A_1} \in \mathcal{O}_p(\overline{A})$.

If $A_1, A_2 \in \mathcal{H}_p(A)$ then $A_1 \vee A_2 \in \mathcal{H}_p(A)$. Thus, if $\overline{A_1} = \overline{A_2}$ then $\overline{A_1} = \overline{A_1 \vee A_2}$ which, by the preservation of the index that was proved in the last paragraph, implies that $A_1 = A_1 \vee A_2$, or $A_2 \leq A_1$. Similarly $A_1 \leq A_2$ and therefore $A_1 = A_2$, that is, the map $A_1 \to \overline{A_1}$ is injective. To see that our application is surjective, for $B \in \mathcal{O}_p(\overline{A})$ we shall see that $B \cap A \in \mathcal{H}_p(A)$ and that $B = \overline{B \cap A}$. In fact, $[\overline{A} : B] = [B(\overline{A \cap N}) : B] = [B(A \cap N) : B] = [A \cap N : A \cap N \cap B] = [A \cap N : (A \cap N) \cap (A \cap B)] = [(A \cap B)(A \cap N) : A \cap B] = [A : A \cap B]$ provided that $(A \cap B)(A \cap N) = A$. It is clear that $(A \cap B)(A \cap N) \leq A$ and since $A \leq \overline{A} = B(\overline{A \cap N}) = B(A \cap N)$, it follows at once that $A \leq (A \cap B)(A \cap N)$. Since $[\overline{A} : B] = [\overline{A \cap N} : B \cap (\overline{A \cap N})]$, which is a power of p, then we obtain that $B \cap A \leq_p A$ and $(A \cap B)(A \cap N) = A$, thus $A \cap B \in \mathcal{H}_p(A)$. Now, the equality $[\overline{A} : B] = [A : A \cap B]$ and the preservation of the index which was proved before imply that $\overline{A \cap B} = B$. Hence we obtain our bijection which is an i'ndex preserving correspondence whose inverse is the map $B \to B \cap A$. Finally, the fact that $\overline{A_1} \cap A = A_1$ implies that A_1 is normal in A if and only if $\overline{A_1}$ is normal in $\overline{A}$. Similarly, if A_1 is also normal in G then $\overline{A_1}$ is normal in G_p, and if A is normal in G and $\overline{A_1}$ is normal in G_p, then $A_1 = \overline{A_1} \cap A$ must be normal in G. □

Corollary 1.2. *The map $A \to \overline{A}$ gives an index-preserving correspondence between $\{A * G : A \leq_p H, AN = H\}$ and $\{B *_f G_p : BN_p = H_p\}$.*

Proof. The sets $\{A * G : A \leq_p H, AN = H\}$ and $\{A \leq_f G_p : AN_p = H_p\}$ are just $\mathcal{H}(A)$ and $\mathcal{O}_p(H)$ respectively. Now apply Proposition 1.1 observing that if H is not normal in G then for $* = \triangleleft$ the two sets are empty. □

Using this corollary, we obtain:

$$\zeta^*_{H,N,p}(s) = \zeta^*_{H_p,N_p}(s) := \sum_{\substack{A *_f G_p \\ AN_p = H_p}} [H_p : A]^{-s}. \tag{1.1}$$

Hence, in order to prove our main theorems we will find a cone integral data $\mathcal{D}^*$ such that for all prime p we have

$$\zeta^*_{H_p,N_p}(s) = (1-p^{-1})^{-h} Z_{\mathcal{D}^*}(s-h-r_1+1,p). \tag{1.2}$$

The method we shall use to express $\zeta^*_{H_p,N_p}(s)$ as a p-adic integral is essentially the same as the one which was used in [dS] to study the zeta functions of compact p-adic analytic groups, which are finite extensions of uniform pro-p groups. Fix a Mal'cev basis $\{x_1,\ldots,x_h\}$ for N and choose a transversal $\{1=\gamma_0,\gamma_1,\ldots,\gamma_{r_1-1},\ldots,\gamma_{r-1}\}$ to the cosets of N in G in such a way that $\{\gamma_0,\ldots,\gamma_{r_1-1}\}$ is a transversal to the cosets of N in H . Let δ be the group operation induced on $\{0,\ldots,r-1\}$ when we identify $\gamma_i N$ with i, that is, $\gamma_i N\gamma_j N = \gamma_{\delta(i,j)}N$.
For a h-tuple $\mathbf{a}=(a_1,\ldots,a_h)\in\mathbb{Z}_p^h$, we write $\mathbf{x}^{\mathbf{a}} = x_1^{a_1}\ldots x_h^{a_h}\in N_p$. It is well known that the map $\varphi:\mathbf{a}\to\mathbf{x}^{\mathbf{a}}$ is a homeomorphism from $\mathbb{Z}_p^h$ onto N_p which preserves measure, that is, such that $\mu(\varphi(A))=\mu(A)$ for all open subset A (here μ is the normalized Haar measure). From this we can obtain the following useful result:

Proposition 1.3. *Let B be an open subgroup of N_p and let $x\in N_p$. Then $\mu(\{\mathbf{a}\in\mathbb{Z}_p^h : \mathbf{x}^{\mathbf{a}}\in xB\}) = [N_p:B]^{-1}$.*

The concept of Mal'cev basis can be also extended to other families of nilpotent groups different than the family of τ-groups. For instance in [W, Chap. 10] the framework to do this is developed. We shall explain quickly without proofs how this is done in the family of finitely generated (topologically) torsion free nilpotent pro-p groups: *τ_p-groups*. Let B a τ_p-group. A Mal'cev basis for B is an ordered set $\{b_1,\ldots,b_h\}$ such that $B=\overline{\langle b_1,\ldots,b_h\rangle}\geq\overline{\langle b_2,\ldots,b_h\rangle}\geq\ldots\geq\overline{\langle b_h\rangle}\geq 1$ is a central series of B with all its factors isomorphic to $\mathbb{Z}_p$. For example in our case $\{x_1,\ldots,x_h\}$ is a Mal'cev basis for the τ_p-group N_p. The existence and the elementary facts about these Mal'cev bases, for example: every element of B can be written in a unique way as a product $b_1^{\lambda_1}\ldots b_h^{\lambda_h}$ with $\lambda_i\in\mathbb{Z}_p$, can be obtained in a similar way as in the case of Mal'cev bases for τ-groups. We shall used them without explicit mention.

It is not difficult to see that for each open subgroup B of N_p, there exists an upper triangular matrix $\mathbf{t}\in Tr(h,\mathbb{Z}_p)$ such that $\{\mathbf{x}^{\mathbf{t}_1},\ldots,\mathbf{x}^{\mathbf{t}_h}\}$ is a Mal'cev basis for B. These bases are called *good bases* for B and we shall say that the matrix $\mathbf{t}$ represents the good basis. Let $\mathcal{M}(B)$ denote the set of all matrices $\mathbf{t}\in Tr(h,\mathbb{Z}_p)$ representing some good basis for the open subgroup B. The following facts are proved in [GSS, Section 2]:

(1) $\mathcal{M}(B)$ is an open subset of $Tr(h,\mathbb{Z}_p)$,

(2) for $\mathbf{t} \in \mathcal{M}(B)$, the norm $|t_{ii}|_p$ depends only on B,
(3) $\mu(\mathcal{M}(B)) = (1-p^{-1})^h \prod_{i=1}^h |t_{ii}|_p^i$, where μ is the Haar measure in $Tr(h, \mathbb{Z}_p)$, and
(4) $[N_p : B] = \prod_{i=1}^h |t_{ii}|_p^{-1}$.

For an open subgroup $A \leq_f G_p$ such that $AN_p = H_p$, it is easy to see that $A = (A \cap N_p) \cup \gamma_1 n_1 (A \cap N_p) \cup \ldots \cup \gamma_{r_1-1} n_{r_1-1}(A \cap N_p)$ for some $n_1, \ldots, n_{r_1-1} \in N_p$, and this allows us to define $\mathcal{T}(A)$ as the set of all pairs of matrices $(\mathbf{t}, \mathbf{v}) \in Tr(h, \mathbb{Z}_p) \times M_{r_1-1\times h}(\mathbb{Z}_p)$ such that $\mathbf{t}$ represents a good basis for $A \cap N_p$ and $\{1, \gamma_1 \mathbf{x}^{\mathbf{v}_1}, \ldots, \gamma_{r_1-1}\mathbf{x}^{\mathbf{v}_{r_1-1}}\}$ is a transversal to the cosets of $A \cap N_p$ in A (here $\mathbf{v}_i$ is the i-th row vector of $\mathbf{v}$).

Lemma 1.4. *$\mathcal{T}(A)$ is an open subset of $Tr(h, \mathbb{Z}_p) \times \mathbb{Z}_p^{h(r_1-1)}$ with Haar measure:*

$$\mu(\mathcal{T}(A)) = (1-p^{-1})^h \prod_{i=1}^h |t_{ii}|_p^{i+r_1-1}.$$

Proof. For $(\mathbf{t}, \mathbf{v}) \in \mathcal{T}(A)$ we clearly have $\mathcal{T}(A) = \mathcal{M}(A \cap N_p) \times \varphi^{-1}(\mathbf{x}^{\mathbf{v}_1}(A \cap N_p)) \times \ldots \times \varphi^{-1}(\mathbf{x}^{\mathbf{v}_{r_1-1}}(A \cap N_p))$, which expresses $\mathcal{T}(A)$ as a product of open subsets. Then the proposition follows from this expression, Proposition 1.3 and the observations above. □

Defining $\mathcal{T}_p^* = \bigcup\{\mathcal{T}(A) : A *_f G_p, AN_p = H_p\}$ and arguing as in the proof of Proposition 2.6 of [GSS], we obtain:

$$\zeta^*_{H_p,N_p}(s) = (1-p^{-1})^{-h} \int_{\mathcal{T}^*} \prod_{i=1}^h |t_{ii}|_p^{s-i-r_1+1} d\mu. \tag{1.3}$$

The next step is to describe $\mathcal{T}_p^{\leq}$ and $\mathcal{T}_p^{\lhd}$ as sets of matrices satisfying cone conditions, that is, we want to find a finite set of polynomials f_i^*, g_i^* with rational coefficients such that for each prime p, $\mathcal{T}_p^* = \{(\mathbf{t}, \mathbf{v}) \in Tr(h, \mathbb{Z}_p) \times M_{r_1-1\times h}(\mathbb{Z}_p) : f_i^*(\mathbf{t}, \mathbf{v}) | g_i^*(\mathbf{t}, \mathbf{v})\}$ up to a set of measure zero. One condition for the pair $(\mathbf{t}, \mathbf{v})$ to be in $\mathcal{T}_p^*$ is that $\mathbf{t}$ must be a good basis for some open subgroup (resp. open normal subgroup) of N_p. We will see that this condition can be described using cone conditions. Once we know that $\mathbf{t}$ represents a good basis for some open subgroup (resp. open normal subgroup) of N_p, then we will see that the other conditions on $(\mathbf{t}, \mathbf{v})$ to be in $\mathcal{T}_p^*$ are a finite number of conditions of the form $\mathbf{x}^{\mathbf{h}(\mathbf{t},\mathbf{v})} \in \overline{\langle \mathbf{x}^{\mathbf{t}_1}, \ldots, \mathbf{x}^{\mathbf{t}_h} \rangle}$ with $\mathbf{h}$ a vectorial polynomial which does not depend on p. Then we have to be able to translate this kind of condition into cone conditions. To do this, we recall the Hall polynomials associated to the Mal'cev basis $\{x_1, \ldots, x_h\}$ of N ([H]). These are polynomials $f_1(\mathbf{X}, \mathbf{Y}), \ldots, f_h(\mathbf{X}, \mathbf{Y}) \in \mathbb{Q}[X_1, \ldots, X_h, Y_1, \ldots, Y_h]$ and

$g_1(\mathbf{X},W),\dots,g_h(\mathbf{X},W)\in\mathbb{Q}[X_1,\dots,X_h,W]$ such that $\mathbf{x}^{\mathbf{a}}\mathbf{x}^{\mathbf{b}}=\mathbf{x}^{\mathbf{f}(\mathbf{a},\mathbf{b})}$ and $(\mathbf{x}^{\mathbf{a}})^w=\mathbf{x}^{\mathbf{g}(\mathbf{a},w)}$ for all $\mathbf{a},\mathbf{b}\in\mathbb{Z}^h$ and $w\in\mathbb{Z}$, where $\mathbf{f}=(f_1,\dots,f_h)$ and $\mathbf{g}=(g_1,\dots,g_h)$.

Now we can describe an algorithm to obtain polynomials j_i,k_i ($i=1,\dots,h$) with rational coefficients such that for all prime p, if $\mathbf{t}$ represents a good basis for some open subgroup of N_p then the condition $\mathbf{x}^{\mathbf{z}}\in\overline{\langle\mathbf{x}^{\mathbf{t}_1},\dots,\mathbf{x}^{\mathbf{t}_h}\rangle}$ is equivalent to $j_i(\mathbf{z},\mathbf{t})|k_i(\mathbf{z},\mathbf{t})$ for all $i=1,\dots,h$:

I. Choose variables T_{ij} for $1\le i\le j\le h$, $Z_1,\dots,Z_h$ and $W_1,\dots,W_h$ and define vectorial polynomials $\mathbf{T}_i=(0,\dots,0,T_{ii},\dots,T_{ih})$ and $\mathbf{Z}=(Z_1,\dots,Z_h)$. For a vectorial polynomial $\mathbf{k}=(k_1,\dots,k_h)$ and $1\le i\le h$, define the vectorial polynomial $\mathbf{k}^i:=(0,\dots,0,k_i,\dots,k_h)$. For example $\mathbf{Z}^2=(0,Z_2,\dots,Z_h)$.

II. Define recursively a list of vectorial polynomials $\mathbf{k}_1,\dots\mathbf{k}_h$ by:
- $\mathbf{k}_1=\mathbf{Z}$,
- $\mathbf{k}_i=\mathbf{f}(\mathbf{g}(\mathbf{g}^i(\mathbf{T}_{i-1},W_{i-1}),-1),\mathbf{k}_{i-1}^i)$ for $1<i\le h$.

Observe that for $i>1$, $\mathbf{k}_i$ is a vectorial polynomial in the variables $\{T_{jl}, j\le l, j<i\}$, $Z_2,\dots,Z_h,W_1,\dots,W_{i-1}$, and for simplicity we shall write $\mathbf{k}_i=\mathbf{k}_i(\mathbf{T}_1,\dots,\mathbf{T}_{i-1},\mathbf{Z}^2,W_1,\dots,W_{i-1})$.

III. Define recursively rational functions $v_1(\mathbf{T},\mathbf{Z}),\dots,v_h(\mathbf{T},\mathbf{Z})$:
- $v_1(\mathbf{T},\mathbf{Z})=k_{11}(Z_1)/T_{11}$
- $v_i(\mathbf{T},\mathbf{Z})=k_{ii}(\mathbf{T}_1,\dots,\mathbf{T}_{i-1},\mathbf{Z}^2,v_1(\mathbf{T},\mathbf{Z}),\dots,v_{i-1}(\mathbf{T},\mathbf{Z}))/T_{ii}$ for $1<i\le h$.

The denominator of these rational functions are all monomials in $T_{11},\dots,T_{hh}$.

Proposition 1.5. *Suppose that $\mathbf{t}\in Tr(h,\mathbb{Z}_p)$ represents a good basis for an open subgroup of N_p and let $\mathbf{z}=(z_1,\dots,z_h)\in\mathbb{Z}_p^h$. Then $x_1^{z_1}\dots x_h^{z_h}\in\overline{\langle x^{\mathbf{t}_1},\dots,x^{\mathbf{t}_h}\rangle}$ if and only if $v_i(\mathbf{t},\mathbf{z})\in\mathbb{Z}_p$ for all i.*

Proof. Let $B_{\mathbf{t}}=\overline{\langle\mathbf{x}^{\mathbf{t}_1},\dots,\mathbf{x}^{\mathbf{t}_h}\rangle}$. Since $\{\mathbf{x}^{\mathbf{t}_1},\dots,\mathbf{x}^{\mathbf{t}_h}\}$ is a Mal'cev basis for $B_{\mathbf{t}}$, then every element of $B_{\mathbf{t}}$ can be written uniquely in the form $(\mathbf{x}^{\mathbf{t}_1})^{a_1}\dots(\mathbf{x}^{\mathbf{t}_h})^{a_h}$ for some $a_i\in\mathbb{Z}_p$. When the element $(\mathbf{x}^{\mathbf{t}_i})^{a_i}\dots(\mathbf{x}^{\mathbf{t}_h})^{a_h}$ is written in the basis $\{x_1,\dots,x_h\}$, it has the form $x_i^{t_{ii}a_i}x_{i+1}^{b_{i+1}}\dots x_h^{b_h}$ for some $b_{i+1},\dots,b_h\in\mathbb{Z}_p$. We shall make use of these facts without mention.

Let $w_i=v_i(\mathbf{t},\mathbf{z})$. We have that $\mathbf{x}^{\mathbf{z}}\in B_t$ if and only if $\mathbf{x}^{\mathbf{z}}=(\mathbf{x}^{\mathbf{t}_1})^{a_1}\dots(\mathbf{x}^{\mathbf{t}_h})^{a_h}$ for some $a_1,\dots,a_h\in\mathbb{Z}_p$, and since $(\mathbf{x}^{\mathbf{t}_1})^{a_1}\dots(\mathbf{x}^{\mathbf{t}_h})^{a_h}$ has the form $x_{11}^{t_{11}a_1}x_2^{b_2}\dots x_h^{b_h}$ then we must have that $t_{11}|z_1$, or $w_1\in\mathbb{Z}_p$, and therefore the element $(\mathbf{x}^{\mathbf{t}_1})^{z_1/t_{11}}=\mathbf{x}^{\mathbf{g}(\mathbf{t}_1,w_1)}$ must be in $B_{\mathbf{t}}$. We conclude that $\mathbf{x}^{\mathbf{z}}\in B_{\mathbf{t}}$ if and only if $w_1\in\mathbb{Z}_p$ and $\mathbf{x}^z=\mathbf{x}^{\mathbf{g}(\mathbf{t}_1,w_1)}a$ for some $a\in B_{\mathbf{t}}$. Now it is easy to see that $g_1(\mathbf{t}_1,w_1)=t_{11}w_1=z_1$, so our condition is $w_1\in\mathbb{Z}_p$ and $x_2^{z_2}\dots x_h^{z_h}=x_2^{g_2(\mathbf{t}_1,w_1)}\dots x_h^{g_d(\mathbf{t}_1,w_1)}a$ for some

$a \in \overline{\langle \mathbf{x}^{\mathbf{t}_2}, \ldots, \mathbf{x}^{\mathbf{t}_h} \rangle}$. Operating with $\mathbf{f}$ and $\mathbf{g}$ we get:

$(x_2^{g_2(\mathbf{t}_1,w_1)} \ldots x_h^{g_h(\mathbf{t}_1,w_1)})^{-1} x_2^{z_2} \ldots x_h^{z_h} = \mathbf{x}^{\mathbf{f}^2(\mathbf{g}^2(\mathbf{g}^2(\mathbf{t}_1,w_1),-1),\mathbf{z}^2)} = \mathbf{x}^{\mathbf{k}_2(\mathbf{t}_1,\mathbf{z}^2,w_1)}$.

Our conclusion is that $\mathbf{x}^{\mathbf{z}} \in B_t$ if and only if $w_1 \in \mathbb{Z}_p$ and $\mathbf{x}^{\mathbf{k}_2(\mathbf{t}_1,\mathbf{z}^2,w_1)} \in \overline{\langle \mathbf{x}^{\mathbf{t}_2}, \ldots, \mathbf{x}^{\mathbf{t}_h} \rangle}$.

Let $i > 1$ and assume that $\mathbf{x}^{\mathbf{z}} \in B_{\mathbf{t}}$ if and only if $w_1, \ldots, w_{i-1} \in \mathbb{Z}_p$ and $\mathbf{x}^{\mathbf{k}_i(\mathbf{t}_1,\ldots,\mathbf{t}_{i-1},\mathbf{z}^2,w_1,\ldots,w_{i-1})} \in \overline{\langle \mathbf{x}^{\mathbf{t}_i}, \ldots, \mathbf{x}^{\mathbf{t}_h} \rangle}$. Working as before and assuming that $w_1, \ldots, w_{i-1} \in \mathbb{Z}_p$, we see that this is equivalent to $t_{ii} | k_{ii}(\mathbf{t}_1, \ldots, \mathbf{t}_{i-1}, \mathbf{z}^2, w_1, \ldots, w_i)$, or $w_i \in \mathbb{Z}_p$, and $\mathbf{x}^{\mathbf{k}_i(\mathbf{t}_1,\ldots,\mathbf{t}_{i-1},\mathbf{z}^2,w_1,\ldots,w_{i-1})} = \mathbf{x}^{\mathbf{g}(\mathbf{t}_i,w_i)} a$ for some $a \in \overline{\langle x_i^{\mathbf{t}_i}, \ldots, x_h^{\mathbf{t}_h} \rangle}$, and again this is equivalent to $w_i \in \mathbb{Z}_p$ and $\mathbf{x}^{\mathbf{k}_i^{i+1}(\mathbf{t}_1,\ldots,\mathbf{t}_{i-1},\mathbf{z}^2,w_1,\ldots,w_i)} = \mathbf{x}^{\mathbf{g}^{i+1}(\mathbf{t}_i,w_i)} a$ for some $a \in \overline{\langle \mathbf{x}^{\mathbf{t}_{i+1}}, \ldots, \mathbf{x}^{\mathbf{t}_d} \rangle}$. The exponent of $(\mathbf{x}^{\mathbf{g}^{i+1}(\mathbf{t}_i,w_i)})^{-1} \mathbf{x}^{\mathbf{k}_i^{i+1}(\mathbf{t}_1,\ldots,\mathbf{t}_{i-1},\mathbf{z}^2,w_1,\ldots,w_i)}$ is clearly

$$\mathbf{f}(\mathbf{g}(\mathbf{g}^{i+1}(\mathbf{t}_i, w_i), -1), \mathbf{k}_i^{i+1}(\mathbf{t}_1, \ldots, \mathbf{t}_{i-1}, \mathbf{z}^2, w_1, \ldots, w_i))$$

which, by definition, is just $\mathbf{k}_{i+1}(\mathbf{t}_1, \ldots, \mathbf{t}_i, \mathbf{z}^2, w_1, \ldots, w_i)$. It follows that $\mathbf{x}^{\mathbf{z}} \in B_{\mathbf{t}}$ if and only if $w_1, \ldots, w_i \in \mathbb{Z}_p$ and $\mathbf{x}^{\mathbf{k}_{i+1}(\mathbf{t}_1,\ldots,\mathbf{t}_i,\mathbf{z}^2,w_1,\ldots,w_i)} \in \overline{\langle \mathbf{x}^{\mathbf{t}_{i+1}}, \ldots, \mathbf{x}^{\mathbf{t}_h} \rangle}$ and then induction applies. □

Corollary 1.6. *There exists a finite number of polynomials j_i, k_i ($i \in I$) with rational coefficients such that for all prime p, the condition for a matrix $\mathbf{t} \in Tr(h, \mathbb{Z}_p)$ to represent a good basis for some open subgroup of N_p is $t_{11} \ldots t_{hh} \neq 0$ and $j_i(\mathbf{t}) | k_i(\mathbf{t})$ for all $i \in I$.*

Proof. This corollary is certainly true if the group N is abelian. Suppose that $h > 1$ and assume that the result is true for all those τ-groups of Hirsch length smaller than h. Then there exists polynomials j_i, k_i ($i \in I_1$) with rational coefficients such that for all prime p, the upper triangular matrix $\mathbf{t}'$ with entries in $\mathbb{Z}_p$ and rows $\mathbf{t}_2, \ldots, \mathbf{t}_h$ represents a good basis for some open subgroup of the pro-p completion of $\langle x_2, \ldots, x_h \rangle$ (which is equal to $\overline{\langle x_2, \ldots, x_h \rangle}$, the closure in N_p) if and only if $t_{22} \ldots t_{hh} \neq 0$ and $j_i(\mathbf{t}') | k_i(\mathbf{t}')$ for all $i \in I_1$. It is shown in [GSS] that a matrix $\mathbf{t} \in Tr(h, \mathbb{Z}_p)$ represents a good basis for some open subgroup of N_p if and only if $\prod_{i=1}^h t_{ii} \neq 0$ and $[\mathbf{x}^{\mathbf{t}_i}, \mathbf{x}^{\mathbf{t}_j}] \in \overline{\langle \mathbf{x}^{\mathbf{t}_{j+1}}, \ldots, \mathbf{x}^{\mathbf{t}_h} \rangle}$ for $1 \leq i < j < h$. Then it is clear that $\{\mathbf{x}^{\mathbf{t}_1}, \ldots, \mathbf{x}^{\mathbf{t}_h}\}$ is a good basis for some open subgroup of N_p if and only if $t_{11} \neq 0$, $[\mathbf{x}^{\mathbf{t}_1}, \mathbf{x}^{\mathbf{t}_j}] \in \overline{\langle \mathbf{x}^{\mathbf{t}_{j+1}}, \ldots \mathbf{x}^{\mathbf{t}_h} \rangle}$ for all $j > 1$ and $\{\mathbf{x}^{\mathbf{t}_2}, \ldots, \mathbf{x}^{\mathbf{t}_h}\}$ is a good basis for some open subgroup of $\overline{\langle x_2, \ldots, x_h \rangle}$. Let $c_1, \ldots, c_h$ be the polynomials in $\mathbb{Q}[X_1, \ldots, X_h, Y_1, \ldots, Y_h]$ such that $[\mathbf{x}^{\mathbf{a}}, \mathbf{x}^{\mathbf{b}}] = \mathbf{x}^{\mathbf{c}(\mathbf{a},\mathbf{b})}$ for all $\mathbf{a}, \mathbf{b} \in \mathbb{Z}^h$. By the proposition above, there exist finite sets of polynomials j_i, k_i, $i \in J_j$, $j = 2, \ldots, h$ with rational coefficients such that for all prime p the following holds: given $\mathbf{t} \in Tr(h, \mathbb{Z}_p)$, if we

assume that $\{\mathbf{x}^{\mathbf{t}_2},\dots,\mathbf{x}^{\mathbf{t}_h}\}$ is a good basis for some open subgroup of $\overline{\langle x_2,\dots,x_h\rangle}$, then for each $j>1$ the condition $\mathbf{x}^{\mathbf{c}(\mathbf{t}_1,\mathbf{t}_j)}\in\overline{\langle \mathbf{x}^{\mathbf{t}_{j+1}},\dots \mathbf{x}^{\mathbf{t}_h}\rangle}$ is equivalent to $j_i(\mathbf{t})|k_i(\mathbf{t})$ for all $i\in J_j$. It follows that the condition for $\mathbf{t}$ to represent a good basis of some open subgroup of N_p is that $t_{11}\dots t_{hh}\neq 0$ and $j_i(\mathbf{t})|k_i(\mathbf{t})$ for all $i\in I_1\cup J_1\cup\dots\cup J_h$. □

Corollary 1.7. *The subgroup zeta function and the normal zeta function of a τ-group are Euler products of cone integrals.*

Proof. For the τ-group N we have that $\zeta^*_N(s)=\prod_p\zeta^*_{N,p}(s)=\prod_p\zeta^*_{N_p}(s)$, and using expression (1.3) we have

$$\zeta^*_N(s)=\prod_p\left((1-p^{-1})^h\int_{\mathcal{T}_p^*}\prod_{i=1}^h|t_{ii}|_p^{s-i}d\mu\right).$$

When $*=\leq$, then Corollary 1.6 implies that the set $\mathcal{T}_p^{\leq}$ can be described by the condition $t_{11}\dots t_{hh}\neq 0$ and a finite set of cone conditions independent of p. Since the set of matrices $\mathbf{t}\in Tr(h,\mathbb{Z}_p)$ with $t_{11}\dots t_{hh}=0$ has Haar measure zero, then we can drop this condition. This proves the corollary in the case $*=\leq$. In the case $*=\lhd$, we have to add some more conditions on the matrix $\mathbf{t}$. It can be checked easily (for instance see [GSS]) that if $\mathbf{t}$ represents a good basis, then the subgroup $B_{\mathbf{t}}$ is normal if and only if $\mathbf{x}^{\mathbf{c}(\mathbf{e}_i,\mathbf{t}_j)}\in\overline{\langle \mathbf{x}^{\mathbf{t}_1},\dots \mathbf{x}^{\mathbf{t}_h}\rangle}$ for all i,j (here $\mathbf{e}_i=(0,\dots,\underbrace{1}_{i},\dots,0)$ and $\mathbf{c}$ is the vectorial polynomial which expresses the commutator). Again, an application of Proposition 1.5 tell us that this condition is equivalent to add cone conditions to the set $\mathcal{T}_p^{\leq}$ where the polynomials are independent of p. □

Proposition 1.8. *There exists a finite set of vectorial polynomials $\{\mathbf{k}_i\}_{i\in I}$ with $k_{ij}\in\mathbb{Q}[\mathbf{T},\mathbf{V}]$ for $j=1,\dots,h$ and $i\in I$, such that for every prime p, the conditions for $(\mathbf{t},\mathbf{v})\in Tr(h,\mathbb{Z}_p)\times M_{r_1-1\times h}(\mathbb{Z}_p)$ to be in $\mathcal{T}_p^*$ are that $\mathbf{t}$ represents a good basis for some open subgroup of N_p and*

$$\mathbf{x}^{\mathbf{k}_i(\mathbf{t},\mathbf{v})}\in\overline{\langle \mathbf{x}^{\mathbf{t}_1},\dots,\mathbf{x}^{\mathbf{t}_h}\rangle},\quad\forall i\in I.$$

Proof. A pair $(\mathbf{t},\mathbf{v})\in\mathrm{Tr}(h,\mathbb{Z}_p)\times M_{r_1-1\times h}(\mathbb{Z}_p)$ is in $\mathcal{T}_p^*$ if and only if $\mathbf{t}$ represents a good basis for some open subgroup $B_{\mathbf{t}}$ of N_p and $A_{(\mathbf{t},\mathbf{v})}:=(B_{\mathbf{t}}\cup\gamma_1\mathbf{x}^{\mathbf{v}_1}B_{\mathbf{t}}\cup\dots\cup\gamma_{r_1-1}\mathbf{x}^{\mathbf{v}_{r_1-1}}B_{\mathbf{t}})*_f G_p$. Then we can assume that $\mathbf{t}$ represents a good basis for the subgroup $B_{\mathbf{t}}$. Recall that δ is the operation on $\{0,\dots,r-1\}$ induced by the quotient G/N when we identify γ_iN with i, and let $\mathbf{x}^{\mathbf{n}_{ij}}\in N$ be such that $\gamma_i\gamma_j=\gamma_{\delta(i,j)}\mathbf{x}^{\mathbf{n}_{ij}}$. It is easy to see then that $A_{(\mathbf{t},\mathbf{v})}$ is a subgroup of G_p if and only if:

(1) $(\gamma_j\mathbf{x}^{\mathbf{v}_j})^{-1}\mathbf{x}^{\mathbf{t}_i}\gamma_j\mathbf{x}^{\mathbf{v}_j}\in B_{\mathbf{t}}$ for $i=1,\dots,h$ and $j=1,\dots,r_1-1$;

(2) for $1 \le i, j \le r_1 - 1$ with $\delta(i,j) \neq 0$, $(\gamma_{\delta(i,j)}\mathbf{x}^{\mathbf{v}_{\delta(i,j)}})^{-1}\gamma_i\mathbf{x}^{\mathbf{v}_i}\gamma_j\mathbf{x}^{\mathbf{v}_j} \in B_{\mathbf{t}}$;
(3) $\gamma_j\mathbf{x}^{\mathbf{v}_j}\gamma_{j^{-1}}\mathbf{x}^{\mathbf{v}_{j^{-1}}} \in B_{\mathbf{t}}$ for $j = 1, \ldots, r_1 - 1$.

In fact, the first condition reflects the normality of B_t in $A_{(\mathbf{t},\mathbf{v})}$ which is necessary since N_p is normal in H_p, and the second and third conditions reflect the fact that $A_{(\mathbf{t},\mathbf{v})}/B_{\mathbf{t}}$ must be a group. If we want $A_{(\mathbf{t},\mathbf{v})}$ to be normal in G_p we have to add more conditions. The first one is that $B_{\mathbf{t}} = N_p \cap A_{(\mathbf{t},\mathbf{v})}$ must be normal and we have to add the conditions $\gamma_i A_{(\mathbf{t},\mathbf{v})}\gamma_i^{-1} = A_{(\mathbf{t},\mathbf{v})}$ for all $i = 1, \ldots, r-1$ and $x_i A_{(\mathbf{t},\mathbf{v})} x_i^{-1} = A_{(\mathbf{t},\mathbf{v})}$ for all $i = 1, \ldots, h$. Assuming that $B_{\mathbf{t}}$ is normal then we have

$$\gamma_i A_{(\mathbf{t},\mathbf{v})}\gamma_i^{-1} = B_{\mathbf{t}} \cup \gamma_i\gamma_1\mathbf{x}^{\mathbf{v}_1}\gamma_i^{-1}B_{\mathbf{t}} \cup \ldots \cup \gamma_i\gamma_{r_1-1}\mathbf{x}^{\mathbf{v}_{r_1-1}}\gamma_i^{-1}B_{\mathbf{t}}$$

and for each j we obtain:

$$\begin{aligned}\gamma_i\gamma_j\mathbf{x}^{\mathbf{v}_j}\gamma_i^{-1} &= \gamma_i\gamma_j\gamma_i^{-1}\gamma_i\mathbf{x}^{\mathbf{v}_j}\gamma_i^{-1}\\ &= \gamma_{\delta(i,j,i^{-1})}\mathbf{x}^{\mathbf{v}_{\delta(i,j,k)}}\big((\mathbf{x}^{\mathbf{v}_{\delta(i,j,i^{-1})}})^{-1}\gamma_{\delta(i,j,i^{-1})}^{-1}(\gamma_i\gamma_j\gamma_i^{-1})(\gamma_i\mathbf{x}^{\mathbf{v}_j}\gamma_i^{-1})\big);\end{aligned}$$

and similarly we have

$$x_i A_{(\mathbf{t},\mathbf{v})} x_i^{-1} = B_{\mathbf{t}} \cup x_i\gamma_1\mathbf{x}^{\mathbf{v}_1}x_i^{-1}B_{\mathbf{t}} \cup \ldots \cup x_i\gamma_{r_1-1}\mathbf{x}^{\mathbf{v}_{r_1-1}}x_i^{-1}B_{\mathbf{t}}$$

and for each j we obtain:

$$x_i\gamma_j\mathbf{x}^{\mathbf{v}_j}x_i^{-1} = \gamma_j\mathbf{x}^{\mathbf{v}_j}\big((\mathbf{x}^{\mathbf{v}_j})^{-1}\gamma_j^{-1}x_i\gamma_j\mathbf{x}^{\mathbf{v}_j}x_i^{-1}\big).$$

Since in the normal case we always assume that H is normal in G then we see that $\{\gamma_{\delta(i,j,i^{-1})}\mathbf{x}^{\mathbf{v}_{\delta(i,j,k)}}N\}_{j=2}^{r_1-1} = \{\gamma_j\mathbf{x}^{\mathbf{v}_j}N\}_{j=2}^{r_1-1}$ and therefore the conditions we have to add in the case $* = \lhd$ are:

(4) $\mathbf{x}^{\mathbf{c}(\epsilon_i,\mathbf{t}_j)} \in B_{\mathbf{t}}$ (remember that this is the condition for $B_{\mathbf{t}}$ to be normal in N_p);
(5) $\gamma_j^{-1}\mathbf{x}^{\mathbf{t}_i}\gamma_j \in B_{\mathbf{t}}$ for $i = 1, \ldots, h$ and $j = 1, \ldots, r-1$;
(6) $(\mathbf{x}^{\mathbf{v}_{\delta(i,j,i^{-1})}})^{-1}\gamma_{\delta(i,j,i^{-1})}^{-1}(\gamma_i\gamma_j\gamma_i^{-1})(\gamma_i\mathbf{x}^{\mathbf{v}_j}\gamma_i^{-1}) \in B_{\mathbf{t}}$ for $i = 1, \ldots, r-1$, $j = 1, \ldots, r_1 - 1$; and
(7) $(\mathbf{x}^{\mathbf{v}_j})^{-1}\gamma_j^{-1}x_i\gamma_j\mathbf{x}^{\mathbf{v}_j}x_i^{-1} \in B_{\mathbf{t}}$, $i = 1, \ldots, h$ and $j = 1, \ldots, r_1 - 1$.

For each $i = 1, \ldots, h$ and $j = 1, \ldots, r-1$, we let $\gamma_j^{-1}x_i\gamma_j = \mathbf{x}^{\mathbf{l}_{ij}}$. For any $\mathbf{u} \in \mathbb{Z}^h$ we have $\gamma_j^{-1}\mathbf{x}^{\mathbf{u}}\gamma_j = (\mathbf{x}^{\mathbf{l}_{1j}})^{u_1}\ldots(\mathbf{x}^{\mathbf{l}_{hj}})^{u_h} = \mathbf{x}^{\mathbf{g}(\mathbf{l}_{1j},u_1)}\ldots\mathbf{x}^{\mathbf{g}(\mathbf{l}_{hj},u_h)} = \mathbf{x}^{\mathbf{f}(\mathbf{f}(\ldots\mathbf{f}(\mathbf{g}(\mathbf{l}_{1j},u_1),\mathbf{g}(\mathbf{l}_{2j},u_2)),\ldots),\mathbf{g}(\mathbf{l}_{hj},u_h))} = \mathbf{x}^{\mathbf{p}_j(\mathbf{u})}$, for some vectorial polynomials $\mathbf{p}_1, \ldots, \mathbf{p}_r$.

Using this, the first condition is equivalent to

$$\mathbf{x}^{\mathbf{f}(\mathbf{f}(\mathbf{g}(\mathbf{v}_j,-1),\mathbf{p}_j(\mathbf{t}_i)),\mathbf{v}_j)} \in \overline{\langle \mathbf{x}^{\mathbf{t}_1}, \ldots, \mathbf{x}^{\mathbf{t}_h}\rangle} \quad j = 1, \ldots, h; i =, \ldots, r_1 - 1.$$

For the second condition, fix a pair $i, j \in \{1, \ldots, r_1 - 1\}$ with $k := \delta(i,j) \neq 0$. Then we have $(\gamma_k\mathbf{x}^{\mathbf{v}_k})^{-1}\gamma_i\mathbf{x}^{\mathbf{v}_i}\gamma_j\mathbf{x}^{\mathbf{v}_j} = \mathbf{x}^{\mathbf{g}(\mathbf{v}_k,-1)}\gamma_k^{-1}\gamma_i\mathbf{x}^{\mathbf{v}_i}\gamma_j\mathbf{x}^{\mathbf{v}_j} =$

$\mathbf{x}^{\mathbf{g}(\mathbf{v}_k,-1)}\mathbf{x}^{\mathbf{n}_{ij}}\gamma_j^{-1}\mathbf{x}^{\mathbf{v}_i}\gamma_j\mathbf{x}^{\mathbf{v}_j} = \mathbf{x}^{\mathbf{g}(\mathbf{v}_k,-1)}\mathbf{x}^{\mathbf{n}_{ij}}\mathbf{x}^{\mathbf{p}_j(\mathbf{v}_i)}\mathbf{x}^{\mathbf{v}_j} = \mathbf{x}^{\mathbf{f}(\mathbf{f}(\mathbf{f}(\mathbf{g}(\mathbf{v}_k,-1),\mathbf{n}_{ij}),\mathbf{p}_j(\mathbf{v}_i)),\mathbf{v}_j)}$, and therefore the second condition is equivalent to

$$\mathbf{x}^{\mathbf{f}(\mathbf{f}(\mathbf{f}(\mathbf{g}(\mathbf{v}_k,-1),\mathbf{n}_{ij}),\mathbf{p}_j(\mathbf{v}_i)),\mathbf{v}_j)} \in \overline{\langle \mathbf{x}^{\mathbf{t}_1}, \dots, \mathbf{x}^{\mathbf{t}_h} \rangle} \quad 2 \le i, j \le r_1,\ \delta(i,k) \neq 0.$$

The other conditions can be handled in the same way. □

We are now in position to prove Theorem A:

Theorem 1.9. *There exists a cone integral data $\mathcal{D}^*$ such that for all primes p we have*

$$\zeta^*_{G,N,p}(s) = (1-p^{-1})^{-h} Z_{\mathcal{D}^*}(s-h-r_1+1,p).$$

Proof. Expression (1.3) gives the expression of $\zeta^*_{H,N,p}(s)$ as an integral, and Corollary 1.6 and Proposition 1.8 gives the description of $\mathcal{T}^*_p$ (up to a set of measure zero) with cone conditions with polynomials independent of p. □

The proof of Theorem B is almost ready from Theorem A and the introduction. We only have to check the bounds for the abscissa of convergence, but this is a particular case of Proposition 5.6.4 from [LS]. There it is considered only the case $* = \le$, but the case $* = \lhd$ can be treated in the same way.

Corollary 1.10. *If $\zeta^*_{H,N}(s)$ has positive abscissa of convergence then the abscissa of convergence of each $\zeta^*_{H,N,p}(s)$ is strictly on the left of the abscissa of convergence of $\zeta^*_{H,N}(s)$. Therefore, if S is a finite set of primes then $\zeta^*_{H,N}(s)$ and $\prod_{p\notin S}\zeta^*_{H,N,p}(s)$ have the same abscissa of convergence.*

Proof. This follows immediately from Theorem A and the properties of cone integrals stated in the Introduction. □

The hypothesis of this corollary are satisfied if we consider the subgroup zeta function:

Proposition 1.11. *The abscissa of convergence of $\zeta^{\le}_{H,N}(s)$ is bounded below by 1.*

Proof. There exists a normal subgroup K of H such that $K \le N$ and H/K has Hirsch length 1. If $N_1 \le N$ is a normal subgroup of H such that $N_1/K \cong \mathbb{Z}$ then $\zeta^{\le}_{H,N_1}(s) \le \zeta^{\le}_{H,N}(s)$ for $s > 0$. Clearly, every subgroup of N_1 containing K is normal in H and therefore for p not dividing the index $[H:N_1]$ and $k \ge 0$ there exists $A \le H$ of index p^k and $AN_1 = H$ (see Hall subgroups for example in [R]). Then for $s > 0$ we have $\zeta^{\le}_{H,N_1}(s) \ge \prod_{p\nmid[H:N_1]}\zeta^{\le}_{\mathbb{Z},p}(s)$ and the last one has abscissa of convergence 1 because $\zeta^{\le}_{\mathbb{Z}}(s) = \zeta(s)$, the Riemann zeta function. □

Corollary 1.12. *Let G' be another group with a finite index normal subgroup N' which is a τ-group. If $\zeta^{\leq}_{G,N,p}(s) = \zeta^{\leq}_{G',N,p}(s)$ for all but a finite number of primes p, then $\zeta^{\leq}_{G,N}(s)$ and $\zeta^{\leq}_{G',N'}(s)$ have the same abscissa of convergence.*

2. Mal'cev completions for virtually nilpotent groups

In this section we introduce the Mal'cev completions of pairs (G, N) with G a finitely generated virtually nilpotent group and N a finite index normal subgroup of G which is a τ-group, and we shall prove Theorem C. This theorem is analogous to the fact that the abscissa of convergence of a τ-group is an invariant of commensurability and therefore it only depends on the $\mathbb{Q}$-Mal'cev completion of the τ-group. Most of the material we shall use is well known but we shall try to do this section self-contained.

2.1. **Mal'cev completions.** Let k be a field of characteristic zero. We fix a pair (G, N) where G is a finitely generated virtually nilpotent group and N is a finite index normal subgroup of G which is a τ-group with k-Mal'cev completion N^k. Choose a transversal $T = \{1, t_2, \ldots, t_r\}$ to the cosets of N in G. The group operation on G/N induces a group operation δ on the set $\{1, \ldots, r\}$ when we identify Nt_i with i. In this way we obtain elements $n_{ij} \in N$ such that $t_i t_j = n_{ij} t_{\delta(i,j)}$. Let σ_i denote the (unique) extension to N^k of the automorphism $n \mapsto n^{t_i} = t_i n t_i^{-1}$ of N. Let K denote the set $N^k \times T$ with the following operation:

$$(x, t_i) * (y, t_j) = (x\sigma_i(y)n_{ij}, t_{\delta(i,j)}) \tag{2.1}$$

Proposition 2.1. *K is a group, $N^k \times \{1\}$ is a normal subgroup of K isomorphic to N^k and the map $\iota : G \to K$ given by $nt_i \to (n, t_i)$ is an injective homomorphism such that $\iota(E) \cap (N^k \times \{1\}) = \iota(N)$ and $\iota(E)(N^k \times \{1\}) = K$. In particular $K/(N^k \times \{1\}) \cong G/N$.*

Proof. Associativity: Let $(x, t_i), (y, t_j), (z, t_k) \in K$. We have

$$\begin{aligned}
((x, t_i) * (y, t_j)) * (z, t_k) &= (x\sigma_i(y)n_{ij}, t_{\delta(i,j)}) * (z, t_k) \\
&= (x\sigma_i(y)n_{ij}\sigma_{\delta(i,j)}(z)n_{\delta(i,j),k}, t_{\delta((\delta(i,j)),k)}) \\
(x, t_i) * ((y, t_j) * (z, t_k)) &= (x, t_i) * (y\sigma_j(z)n_{jk}, t_{\delta(j,k)}) \\
&= (x\sigma_i(y\sigma_j(z)n_{jk})n_{i,\delta(j,k)}, t_{\delta(i,\delta(j,k))})
\end{aligned}$$

Now it is clear that we only have to check that $n_{ij}\sigma_{\delta(i,j)}(z)n_{\delta(i,j),k} = \sigma_i(\sigma_j(z)n_{jk})n_{i,\delta(j,k)}$ for all $z \in N^k$. Of course it will be enough to check this just for $z \in N$. In this case we have

$$n_{ij}\sigma_{\delta(i,j)}(z)n_{\delta(i,j),k} = t_i t_j t^{-1}_{\delta(i,j)} t_{\delta(i,j)} z t^{-1}_{\delta(i,j)} t_{\delta(i,j)} t_k t^{-1}_{\delta(\delta(i,j),k)} = t_i t_j z t_k t^{-1}_{\delta(\delta(i,j),k)}$$

and

$$\sigma_i(\sigma_j(z)n_{jk})n_{i,\delta(j,k)} = t_it_jzt_j^{-1}t_jt_kt_{\delta(j,k)}^{-1}t_i^{-1}t_it_{\delta(j,k)}t_{\delta(i,\delta(j,k))}^{-1} = t_it_jzt_kt_{\delta(i,\delta(j,k))}^{-1}$$

which are equal.

Unit and existence of the inverse: It is clear that the element $(1,1)$ works as a unit and for (x,t_i) we shall see that $(\sigma_i^{-1}(x^{-1}n_{i,i^{-1}}^{-1}), t_{i^{-1}})$ works as its inverse. In fact,

$$(x,t_i) * (\sigma_i^{-1}(x^{-1}n_{i,i^{-1}}^{-1}), t_{i^{-1}}) = (x\sigma_i(\sigma_i^{-1}(x^{-1}n_{i,i^{-1}}^{-1}))n_{i,i^{-1}}, 1) = (1,1)$$

and

$$(\sigma_i^{-1}(x^{-1}n_{i,i^{-1}}^{-1}), t_{i^{-1}}) * (x,t_i) = (\sigma_i^{-1}(x^{-1}n_{i,i^{-1}}^{-1})\sigma_{i^{-1}}(x)n_{i^{-1},i}, 1)$$

and if we want to prove that $\sigma_i^{-1}(x^{-1}n_{i,i^{-1}}^{-1})\sigma_{i^{-1}}(x)n_{i^{-1},i} = 1$ for all $x \in N^k$ it will be enough to do it for all $x \in N$. In this case we have

$$\begin{aligned}\sigma_i^{-1}(x^{-1}n_{i,i^{-1}}^{-1})\sigma_{i^{-1}}(x)n_{i^{-1},i} &= t_i^{-1}x^{-1}n_{i,i^{-1}}^{-1}t_it_{i^{-1}}xt_{i^{-1}}^{-1}n_{i^{-1},i}\\ &= t_i^{-1}x^{-1}t_{\delta(i,i^{-1})}xt_{i^{-1}}^{-1}n_{i^{-1},i} = t_i^{-1}t_{i^{-1}}^{-1}n_{i^{-1},i} = t_{i^{-1},i} = 1\end{aligned}$$

Then K is a group and it follows from the definition of the operation that $N^k \times \{1\}$ is a subgroup of K and that the map $x \to (x,1)$ gives an isomorphism from N^k onto $N^k \times \{1\}$. Since $(x,t_i) = (x,1)*(1,t_i)$ and $(1,t_i)*(x,1)*(1,t_i)^{-1} = (\sigma_i(x),t_i)*(\sigma_i^{-1}(n_{i,i^{-1}}^{-1}),t_{i^{-1}}) = (\sigma_i(x),1)$, then $N^k \times \{1\}$ is normal in K. Finally define $\iota : G \to K$ by $nt_i \to (n,t_i)$. Since $nt_imt_j = nt_imt_i^{-1}t_it_j = n\sigma_i(m)n_{ij}t_{\delta(i,j)}$ then ι is a homomorphism which is clearly injective and the fact that $(x,1)*(1,t_i) = (x,t_i)$ implies that $(N^k \times \{1\})\iota(G) = K$ and clearly $\iota(G) \cap (N^k \times \{1\}) = N \times \{1\} = \iota(N)$. □

Definition 2.2. For a pair (G,N), where G is a group and N a finite index normal subgroup of G which is a τ-group, we define the k-Mal'cev completion of (G,N) as a triple (K,M,ι), where K is a group, M a normal subgroup of K isomorphic to the k-Mal'cev completion of N and $\iota : G \to K$ is an injective homomorphism such that $\iota(G)M = K$ and $\iota(G) \cap M = \iota(N)$.

By the proposition above, any such a pair (G,N) has a k-Mal'cev completion, and it follows from the definition that for any k-Mal'cev completion (K,M,ι) of (G,N) we have $K/M \cong G/N$. In particular, M has finite index in K. Since the k-Mal'cev completion of N is a radicable group then if (K_1,M_1,ι_1) and (K_2,M_2,ι_2) are two k-Mal'cev completions of (G,N) then an isomorphism $\varphi : K_1 \to K_2$ (if there exists) must induce an isomorphism $\varphi|_{M_1} : M_1 \to M_2$. In fact, $\varphi(M_1) \cap M_2$ must have finite index in M_2 and therefore it must be equal to M_2

and similarly M_2 has finite index in $\varphi(M_1)$ and therefore it must be equal to $\varphi(M_1)$.

Proposition 2.3. *Let (K_1, M_1, ι_1) and (K_2, M_2, ι_2) be two k-Mal'cev completions of the pair (G, N). Then there exists a unique isomorphism $\varphi : K_1 \to K_2$ such that the following diagram*

$$\begin{tikzcd}
M_1 \arrow[rrr, hook] \arrow[dd, "\psi"'] & & & K_1 \arrow[dd, "\varphi"] \\
& N \arrow[ul, "\iota_1"'] \arrow[dl, "\iota_2"] \arrow[r, hook] & G \arrow[ur, "\iota_1"] \arrow[dr, "\iota_2"'] & \\
M_2 \arrow[rrr, hook] & & & K_2
\end{tikzcd}$$

commutes (here ψ is the extension of the isomorphism $\iota_2 \circ \iota_1^{-1} : \iota_1(N) \to \iota_2(N)$).

Proof. Take a transversal $T = \{1, t_2, \ldots, t_r\}$ to the cosets of N in G and define δ in $\{1, \ldots, r\}$ and $n_{ij} \in N$ as we did at the beginning of the section. It is clear from the definition that every element of K_i can be written in a unique way as a product $x\iota_i(t_k)$ with $x \in M_i$ and $t_k \in T$. Then, if such a φ exits then it must be defined by $\varphi(x\iota_1(t_k)) = \psi(x)\iota_2(t_k)$. Thus it is enough to check that if we define φ in this way then it is a homomorphism which would be automatically an isomorphism with the required properties.

At one hand we have

$$\begin{aligned}
\varphi(x\iota_1(t_i)y\iota_1(t_j)) &= \varphi(xy^{\iota_1(t_i)}\iota_1(t_it_j)) = \varphi(xy^{\iota_1(t_i)}\iota_1(n_{ij}t_{\delta(i,j)})) \\
&= \psi(xy^{\iota_1(t_i)}\iota_1(n_{ij}))\iota_2(t_{\delta(i,j)}) = \varphi(x)\varphi(y^{\iota_1(t_i)})\iota_2(n_{ij})\iota_2(t_{\delta(i,j)})
\end{aligned}$$

On the other hand we have

$$\begin{aligned}
\varphi(x\iota_1(t_i))\varphi(y\iota_1(t_j)) &= \psi(x)\iota_2(t_i)\psi(y)\iota_2(t_j) = \psi(x)\psi(y)^{\iota_2(t_k)}\iota_2(t_it_j) \\
&= \psi(x)\psi(y)^{\iota_2(t_i)}\iota_2(n_{ij})\iota_2(t_{\delta(i,j)})
\end{aligned}$$

and then it is enough to check that $\psi(x^{\iota_1(t_k)}) = \psi(x)^{\iota_2(t_k)}$ for all $x \in M_1$. But again it is enough to check this equality for all x in $\iota_1(N)$ because M_1 is the k-Mal'cev completion of $\iota_1(N)$. In this case the first member is equal to $\psi(\iota_1(t_k)n\iota_1(t_k)^{-1}) = \psi(\iota(t_knt_k^{-1})) = \iota_2(t_knt_k^{-1})$ and the second member is equal to $\iota_2(t_k)\psi(\iota_1(n))\iota_2(t_k)^{-1} = \iota_2(t_k)\iota_2(n)\iota_2(t_k^{-1}) = \iota_2(t_knt_k^{-1})$ and the proof is complete. $\square$

The k-Mal'cev completion of (G, N) will be denoted simply by (K, M), or K, for the k-Mal'cev completion of (G, N) and it will be understood that G is a subgroup of K such that $GM = K$ and $G \cap M = N$ (M

is the unique finite index subgroup of K which is isomorphic to the k-Mal'cev completion of N). Observe that K depends on both N and G. For example, if G is a finitely generated torsion free nilpotent group and N is a proper finite index subgroup of G then K will have a proper finite index subgroup isomorphic to the k-Mal'cev completion of G (or N) whereas if $N = G$ then $K = N^k$ which has not proper finite index subgroups.

2.2. **The abscissa of convergence as an invariant of the Mal'cev completion.** Let K be a group with a finite index normal subgroup M which is the $\mathbb{Q}$-Mal'cev completion of a τ-group. We denote by $\mathcal{H}(K)$ the family of all subgroups G of K such that $GM = K$ and $G \cap M$ is a τ-group which has M as its $\mathbb{Q}$-Mal'cev completion. It follows from the definitions that for each $G \in \mathcal{H}(K)$, the group K is the $\mathbb{Q}$-Mal'cev completion of the pair $(G, M \cap G)$ and that if (G', N') is any pair with $\mathbb{Q}$-Mal'cev completion isomorphic to K, then there exists an isomorphism from G' onto some $G \in \mathcal{H}(K)$ sending N' onto $G \cap M$. Theorem C will follows from the following proposition:

Proposition 2.4. *If $G_1, G_2 \in \mathcal{H}(K)$ then $\zeta^{\leq}_{G_1, G_1 \cap M}(s)$ and $\zeta^{\leq}_{G_2, G_2 \cap M}(s)$ have the same abscissa of convergence.*

Proof. If $G_1, G_2 \in \mathcal{H}(K)$ then they clearly finitely generated and hence $G_1 \vee G_2$ is also finitely generated. Then $(G_1 \vee G_2) \cap M$ is finitely generated and therefore it is a τ-group whose $\mathbb{Q}$-Mal'cev completion is clearly M. Then $G_1 \vee G_2 \in \mathcal{H}(K)$ and hence we can assume $G_2 \leq G_1$. By Corollary 1.12 it is enough to prove that $\zeta^{\leq}_{G_1, G_1 \cap M}(s)$ and $\zeta^{\leq}_{G_2, G_2 \cap M}(s)$ have the same local factors for all but a finite number of primes. Choose a prime p which does not divide $[G_1 : G_2]$ and consider the family $\mathcal{N}_p$ of all normal subgroups of G_1 contained in $G_1 \cap M$ and which are of index a power of p in $G_1 \cap M$ and consider G_{1p}, the profinite completion of G_1 with respect to this family. Then by Proposition 1.1, the closure map gives a correspondence between the subgroups A of G_i of index a power of p which satisfy $A(G_i \cap M) = G_i$ and the open subgroups B of $\overline{G_i}$ for which $B(\overline{M \cap G_i}) = \overline{G_i}$. Then it is enough to see that $\overline{M \cap G_1} = \overline{M \cap G_2}$ and $\overline{G_1} = \overline{G_2}$. The first equality follows because $[M \cap G_1 : M \cap G_2] = [G_1 : G_2]$ which is not divisible by p and therefore $M \cap G_1$ and $M \cap G_2$ must have the same closure since $M \cap G_2$ is a τ-group. For the second equality we have $\overline{G_1} = \overline{G_2(M \cap G_1)} = G_2\overline{M \cap G_1} = G_2\overline{M \cap G_2} = \overline{G_2}$. ☐

Corollary 2.5. *For each $M \leq L \leq K$ there exists a rational number α_L such that for any $H \in \mathcal{H}(L)$, the zeta function $\zeta^{\leq}_{H, H \cap M}(s)$ has abscissa of convergence α_L.*

Proof. This follows immediately from the proposition above, Theorem A and the fact that global cone integrals have rational abscissa of convergence (see Introduction). □

2.3. **Criterion for having the same Mal'cev completion.** It is useful to have a criterion to decide when two pairs (G, N) and (G', N') have the same $\mathbb{Q}$-Mal'cev completion and conclude in this way that $\zeta_{G,N}(s)$ and $\zeta_{G',N'}(s)$ have the same abscissas of convergence. We shall give such a criterion in Proposition 2.9, but first we shall introduce some definitions in a more general setting.

Consider the family of all pairs (A, B) where A is a group and B is a normal subgroup of A. We say that two pairs (A_1, B_1) and (A_2, B_2) are equivalent, and we write $(A_1, B_1) \sim (A_2, B_2)$, if there exists a group F, two short exact sequences $1 \to B_i \hookrightarrow A_i \xrightarrow{p_i} F \to 1$ and an isomorphism $\psi : B_1 \to B_2$ such that if $\varphi_i : F \to Out(B_i)$ are the maps induced by these exact sequences and $\bar{\psi} : Out(B_1) \to Out(B_2)$ is the isomorphism induced by ψ, then $\bar{\psi} \circ \varphi_1 = \varphi_2$. Before proving that this relation is an equivalence relation we shall need the following lemma:

Lemma 2.6. *Suppose that $1 \to B \hookrightarrow A \xrightarrow{p_1} F_1 \to 1$ and $1 \to B \hookrightarrow A \xrightarrow{p_2} F_2 \to 1$ are two short exact sequences. Then there exists a unique isomorphism $\Phi : F_1 \to F_2$ such that $\Phi \circ p_1 = p_2$. Moreover, if $\varphi_i : F_i \to Out(B)$ are the respective induced maps by these sequences then $\varphi_1 = \varphi_2 \circ \Phi$.*

Proof. Let $\{a_f\}_{f \in F_1}$ be a family of elements in A such that $p_1(a_f) = f$. Define $\Phi(f) = p_2(a_f)$. This is a good definition because $\ker p_1 = \ker p_2 = B$ and it is a homomorphism since for $f, f' \in F$ we have $a_f a_{f'} a_{ff'}^{-1} \in B$ and therefore $p_2(a_f)p_2(a_{f'}) = p_2(a_{ff'})$, that is, $\Phi(f)\Phi(f') = \Phi(ff')$. It is clear that Φ is an isomorphism which is unique with the required property. Finally, if σ_{a_f} denotes conjugation by a_f in B then the image of f by φ_1 is $\sigma_{a_f} Inn(B)$ while, since $p_2(a_f) = \Phi(f)$, the image of $\Phi(f)$ by φ_2 is also $\sigma_{a_f} Inn(B)$. That is, $\varphi_2 \circ \Phi = \varphi_1$. □

Proposition 2.7. *The relation $\sim$ is an equivalence relation.*

Proof. Reflexivity and symmetry are immediate. To prove transitivity let $(A_1, B_1) \sim (A_2, B_2)$ and $(A_2, B_2) \sim (A_3, B_3)$. Let F_1 and F_2 be groups, $1 \to B_1 \hookrightarrow A_1 \xrightarrow{p_1} F_1 \to 1$, $1 \to B_2 \hookrightarrow A_2 \xrightarrow{p_2} F_1 \to 1$, $1 \to B_2 \hookrightarrow A_2 \xrightarrow{q_2} F_2 \to 1$ and $1 \to B_3 \hookrightarrow A_3 \xrightarrow{q_3} F_2 \to 1$ short exact sequences with induced maps $\varphi_i : F_1 \to Out(B_i)$ $(i = 1, 2)$ and $\phi_i : F_2 \to Out(B_i)$ $(i = 2, 3)$, and isomorphisms $\psi_{12} : B_1 \to B_2$ and $\psi_{23} : B_2 \to B_3$ such that $\bar{\psi}_{12} \circ \varphi_1 = \varphi_2$ and $\bar{\psi}_{23} \circ \phi_2 = \phi_3$. The

isomorphism $\Phi : F_1 \to F_2$ of the lemma above satisfies $\Phi \circ p_2 = q_2$ and therefore $\phi_2 \circ \Phi = \varphi_2$. Consider the following two exact sequences

$$1 \to B_1 \hookrightarrow A_1 \xrightarrow{p_1} F_1 \to 1$$

$$1 \to B_3 \hookrightarrow A_3 \xrightarrow{\Phi^{-1}\circ q_3} F_1 \to 1$$

and the isomorphism $\psi = \psi_2 \circ \psi_1 : B_1 \to B_3$. If $\tilde{\varphi} : F_1 \to Out(B_3)$ is the homomorphism induced by the second exact sequence then we have to prove that $\bar{\psi} \circ \varphi_1 = \tilde{\varphi}$. Since $\Phi(\Phi^{-1} \circ q_3) = q_3$, an application of the lemma again implies that $\tilde{\varphi} = \phi_3 \circ \Phi$. Then what we have to show is the equality $\bar{\psi} \circ \varphi_1 = \phi_3 \circ \Phi$. But $\bar{\psi} \circ \varphi_1 = \bar{\psi}_{23} \circ \bar{\psi}_{12} \circ \varphi_1 = \bar{\psi}_{23} \circ \varphi_2 = \bar{\psi}_{23} \circ \phi_2 \circ \Phi = \phi_3 \circ \Phi$ □

Observe that if $(A_1, B_1) \sim (A_2, B_2)$ then $B_1 \cong B_2$ and $A_1/B_1 \cong A_2/B_2$ but it is not true in general that A_1 and A_2 are isomorphic. However we have the following:

Proposition 2.8. *If $(K_1, M_1) \sim (K_2, M_2)$ where M_1 is a finite dimensional k-radicable nilpotent group for some field k of characteristic zero and if K_1/M_1 is finite then K_1 and K_2 are isomorphic.*

Proof. There exist exact sequences $1 \to M_i \hookrightarrow K_i \to F \to 1$ $(i = 1, 2)$ giving homomorphisms $\varphi_i : F \to Out(M_i)$ and an isomorphism $\psi : M_1 \to M_2$ such that if $\bar{\psi} : Out(M_1) \to Out(M_2)$ is the isomorphism induced by ψ then $\bar{\psi} \circ \varphi_1 = \varphi_2$. Let $\pi_i : Aut(M_i) \to Out(M_i)$ the quotient map and let also consider $\tilde{\psi} : Aut(M_1) \to Aut(M_2)$, the map induced by ψ. Let $\tilde{\varphi}_1 : F \to Aut(M_1)$ a lifting of φ_1 ([D, Lemma 3.1.2]) and define $\tilde{\varphi}_2 := \tilde{\psi}\circ\tilde{\varphi}_1$. Since $\pi_2\circ\tilde{\psi} = \bar{\psi}\circ\pi_1$ then $\pi_2\circ\tilde{\varphi}_2 = \pi_2\circ\tilde{\psi}\circ\tilde{\varphi}_1 = \bar{\psi} \circ \pi \circ \tilde{\varphi}_1 = \bar{\psi} \circ \varphi_1 = \varphi_2$ and therefore $\tilde{\varphi}_2$ is a lifting of φ_2. By [D, Lemma 3.1.2] we have that $K_1 \cong M_1 \rtimes_{\tilde{\varphi}_1} F$ and $K_2 \cong M_2 \rtimes_{\tilde{\varphi}_2} F$. Finally define $h : M_1 \rtimes_{\tilde{\varphi}_1} F \to M_2 \rtimes_{\tilde{\varphi}_2} F$ by $h(g, f) = (\psi(g), f)$. This map is clearly bijective and at one hand we have

$$\begin{aligned} h((g_1, f_1) \cdot (g_2, f_2)) &= h(g_1\tilde{\varphi}_1(f_1)(g_2), f_1f_2) = (\psi(g_1\tilde{\varphi}_1(f_1)(g_2)), f_1f_2) \\ &= (\psi(g_1)\psi(\tilde{\varphi}_1(f_1)(g_2)), f_1f_2). \end{aligned}$$

and on the other hand we have

$$(\psi(g_1), f_1)(\psi(g_2), f_2) = (\psi(g_1)\tilde{\varphi}_2(f_1)(\psi(g_2)), f_1f_2)$$

and then all we only have to prove that $\psi(\tilde{\varphi}_1(f)(g)) = \tilde{\varphi}_2(f)(\psi(g))$ for all $f \in F$ and $g \in M_1$. By definition of $\tilde{\varphi}_2$ and $\tilde{\psi}$ we have $\tilde{\varphi}_2(f) = \tilde{\psi} \circ \varphi_1(f) = \psi \circ \tilde{\varphi}_1 \circ \psi^{-1}$. Evaluating this equality in $\psi(g)$ we obtain $\tilde{\varphi}_2(f)(\psi(g)) = \psi \circ \tilde{\varphi}_1(g)$ and this is what we wanted to prove. □

Proposition 2.9. *Suppose that* $(G_1, N_1) \sim (G_2, N_2)$ *where* N_1 *is a* τ-*group and* G_1/N_1 *is finite. If* (K_i, M_i) *is the k-Mal'cev completion of* (G_i, N_i) *then* $(K_1, M_1) \sim (K_2, M_2)$ *and therefore* $K_1 \cong K_2$.

Proof. Let $1 \to N_i \hookrightarrow G_i \xrightarrow{p_i} F \to 1$ $(i = 1, 2)$ with induced maps $\varphi_i : F \to Out(N_i)$ and let $\psi : N_1 \to N_2$ be the isomorphism such that $\bar{\psi} \circ \varphi_1 = \varphi_2$. Let ψ also denote the unique extension $\psi : M_1 \to M_2$ of ψ. We shall consider the isomorphisms $G_i/N_i \cong K_i/M_i$ induced by the inclusion map $G_i \hookrightarrow K_i$ and we denote by q_i the composition $K_i \to K_i/M_i \cong G_i/N_i \xrightarrow{p_i} F$. Then we have exact sequences $1 \to M_i \to K_i \xrightarrow{p_i} F \to 1$ with induced maps, say $\tilde{\varphi}_i : F \to Out(M_i)$ $(i = 1, 2)$. We have to prove that $\bar{\psi} \circ \tilde{\varphi}_1 = \tilde{\varphi}_2$.

Let $f \in F$ and pick $g_i \in G_i$ such that $p_i(g_i) = f$. It follows that $q_i(g_i) = f$ and if $\sigma_i : M_i \to M_i$ denotes conjugation by g_i then we have to show that $\bar{\psi}(\sigma_1 Inn(M_1)) = \sigma_2 Inn(M_2)$, or equivalently, that $\psi \circ \sigma_1 \circ \psi^{-1} \equiv \sigma_2 \mod Inn(M_2)$. If $Res\sigma_i$ is the restriction of σ_i to N_i then by hypothesis we have $\psi \circ Res\sigma_1 \circ \psi^{-1} \equiv Res\sigma_2 \mod Inn(N_2)$, that is, there exists $m \in N_2$ such that $\sigma_2^{-1} \circ \psi \circ \sigma_1 \circ \psi^{-1}(n) = n^m$ for all $n \in N_2$. But then this is true for all $n \in M_2$ and therefore $\psi \circ \sigma_1 \circ \psi^{-1} \equiv \sigma_2 \mod Inn(M_2)$ which is what we wanted to prove. □

3. Abscissa of convergence of zeta functions of virtually abelian groups

Let G be a finitely generated virtually abelian group an let N be a normal subgroup of G isomorphic to $\mathbb{Z}^h$. In [dSMS] it is given a theoretical procedure to obtain explicitly $\zeta^{\leq}_{G,N}(s)$ up to a finite number of local factors. By Corollary 1.12, this is enough to obtain the abscissa of convergence $\alpha^{\leq}(G, N)$ of $\zeta^{\leq}_{G,N}(s)$. The procedure is:

(1) Let $P = G/N$. The $\mathbb{Q}$-algebra $A = \mathbb{Q}P$ decomposes into a product of simple algebras $A = A_1 \times \ldots \times A_r$.
(2) If K_i is the center of A_i with ring of integers R_i, then A_i is isomorphic to a full ring of matrices of rank m_i over some central K_i-division algebra D_i.
(3) Put $n_i^2 = \dim_{K_i}(A_i) = m_i^2 e_i^2$ where $e_i^2 = dim_{K_i}(D_i)$.
(4) The group P acts on N by conjugation. The A-module $V = N \otimes \mathbb{Q}$ decomposes into a direct sum $V = V_1 \oplus \ldots \oplus V_r$ with $V_i = A_i V$ and $V_i = W_i^{k_i}$ where $W_i = D_i^{m_i}$.
(5) Set $\epsilon_i = 0$ or 1 according to whether $C_{W_i}(P) = W_i$ or 0, where $C_{W_i}(P) = \{x \in W_i : x^g = x, \forall g \in P\}$.

(6) The zeta function $\zeta^{\leq}_{G,N}(s)$ and the following zeta function:

$$\prod_{i=1}^{r}\prod_{j=0}^{k_ie_i-1}\zeta_{R_i}(n_i(s-\epsilon_i)-j),$$

where $\zeta_{R_i}(s)$ is the Dedekind zeta function of R_i, have the same local factors for all but a finite number of primes.

Then $\alpha^{\leq}(G,N)$ will be the same as the abscissa of convergence of $\prod_{i=1}^{r}\prod_{j=0}^{k_ie_i-1}\zeta_{R_i}(n_i(s-\epsilon_i)-j)$. Since the Dedekind zeta function of any number field has abscissa of convergence 1 then we have

$$\begin{aligned}\alpha^{\leq}(G,N) &= \max\{\frac{1+j}{n_i}+\epsilon_i : 1\leq i\leq r, 0\leq j\leq k_ie_i-1\}\\ &= \max\{\frac{k_ie_i}{n_i}+\epsilon_i : 1\leq i\leq r\}\\ &= \max\{\frac{k_i}{m_i}+\epsilon_i : 1\leq i\leq r\} \qquad (3.1)\end{aligned}$$

The abscissa of convergence $\alpha^{\leq}(G)$ of $\zeta^{\leq}_{G}(s)$ is not such an interesting invariant. In fact, from the expression $\zeta^{\leq}_{G}(s)=\sum_{N\leq H\leq G}[G:H]^{-s}\zeta^{\leq}_{H,N}(s)$ we see that $\alpha^{\leq}(G)$ is the maximum of the $\alpha^{\leq}(H,N)$ with $N\leq H\leq G$ and since $\alpha^{\leq}(N,N)=h$ then to obtain $\alpha^{\leq}(G)$ different from h we must find $N\leq H\leq G$ such that $\alpha^{\leq}(H,N)>h$. Now observe that

$$h=\sum_{i=1}^{r}\dim_{\mathbb{Q}}V_i=\sum_{i=1}^{r}k_i\dim_{\mathbb{Q}}W_i=\sum_{i=1}^{r}k_im_i\dim_{\mathbb{Q}}D_i \qquad (3.2)$$

Looking at (3.1), we are interested in the case where there exist i such that $\frac{k_i}{m_i}+\epsilon_i>h$. If $k_i<h$ then $\frac{k_i}{m_i}+\epsilon_i\leq h$. Then we are interested in the case where there exists i_0 such that $k_{i_0}=h$ and $\epsilon_{i_0}=1$. By (3.2), we must have $m_{i_0}=1$ and $k_j=0$ for all $j\neq i_0$. Since $V=V_{i_0}^h=D_{i_0}^h$, then $D_{i_0}=\mathbb{Q}$, $A_{i_0}\cong\mathbb{Q}$, $A_{i_0}V=V$ and the fact that $\epsilon_i=1$ says that P has no non-trivial fixed points in V. Then

Corollary 3.1. *If $\alpha^{\leq}(G)$ is either h or $h+1$. The later is the case if and only if there exist an intermediate subgroup $N\leq H\leq G$ such that if $P=H/N$ then $\mathbb{Q}P$ has a simple two-side ideal A_1 isomorphic to $\mathbb{Q}$ such that $A_1(N\otimes\mathbb{Q})=N\otimes\mathbb{Q}$ and P has no non-trivial fixed points in N.*

4. Zeta function of 3-dimensional almost Bieberbach groups

In this section we present the explicit zeta functions of all finitely generated torsion free virtually nilpotent groups of Hirsch length 3. This family of groups is the same as the family of the 3-dimensional almost Bieberbach groups (abbreviated AB-groups), that is, finitely generated torsion free nilpotent groups G such that the maximal normal nilpotent subgroup, the Fitting subgroup $Fitt(G)$, is maximal between all the finite index nilpotent subgroups of G and $h(G) = 3$. We only present the results. The interested reader should consult [S] for the details of the computation. When $Fitt(G)$ is abelian, then we have a 3-dimensional Bieberbach group and a complete list of them (there are 10) can be found in [LSY]. A complete classification of the 3-dimensional AB-group with non-abelian Fitting subgroup can be found in [D] or [DIKL].

It is an important fact that if G is an AB-group with Fitting subgroup N, and if $N \leq H \leq G$, then H is also an AB-group with Fitting subgroup N. Hence we shall only present the zeta functions of pairs $\zeta^{\leq}_{G,N}(s)$, where G is an AB-group and N its Fitting subgroup.

4.1. **The 3-dimensional Bieberbach groups.**

In the table, we present the 3-dimensional Bieberbach groups numbered according to the International Table for Crystallography (IT), we present $\zeta^{\leq}_{G,N}(s)$, where N is the Fitting subgroup of G, its abscissa of convergence and a functional equation which is satisfied for almost all its local factors. In contrast with the situation for τ-groups (see [V]), these functional equation will encode more information than only the Hirsch length of G.

G	$\zeta^{\leq}_{G,N}(s)$	$\alpha(G,N)$	Functional equation
$\mathbb{Z}^3$	$\zeta(s)\zeta(s-1)\zeta(s-2)$	3	$\zeta_{G,p}(s)\lvert_{p\to p^{-1}} = (-1)^{-3}p^{-3s+3}\zeta_{G,p}(s)$
IT=(3,4)	$\zeta(s)\zeta(s-1)\zeta(s-2)$	3	$\zeta^{\leq}_{G,N,p}(s)\lvert_{p\to p^{-1}} = (-1)^{3}p^{-3s+3}\zeta^{\leq}_{G,N,p}(s)$
IT=(3,144)	$\zeta(s)\zeta(s-1)L(s-1,\chi_3)\frac{1}{\zeta_3(s)}$	2	$\zeta^{\leq}_{G,N,p}(s)\lvert_{p\to p^{-1}} = (-1)^{3}\chi_3(p)p^{-3s+2}\zeta^{\leq}_{G,N,p}(s)$
IT=(3,76)	$\zeta(s)\zeta(s-1)L(s-1,\chi_4)\frac{1}{\zeta_2(s)}$	2	$\zeta^{\leq}_{G,N,p}(s)\lvert_{p\to p^{-1}} = (-1)^{3}\chi_4(p)p^{-3s+2}\zeta^{\leq}_{G,N,p}(s)$
IT=(3,169)	$\zeta(s)\zeta(s-1)L(s-1,\chi_3)\frac{1}{\zeta_2(s)\zeta_3(s)}$	2	$\zeta^{\leq}_{G,N,p}(s)\lvert_{p\to p^{-1}} = (-1)^{3}\chi_3(p)p^{-3s+2}\zeta^{\leq}_{G,N,p}(s)$
IT=(3,19)	$\zeta(s-1)^3\zeta_2(s-1)^{-3}$	2	$\zeta^{\leq}_{G,N,p}(s)\lvert_{p\to p^{-1}} = (-1)^{3}p^{-3s+3}\zeta^{\leq}_{G,N,p}(s)$
IT=(3,7)	$\zeta(s-1)^2\zeta(s)\zeta_2(s)^{-1}(1+2^{1-s})$	2	$\zeta^{\leq}_{G,N,p}(s)\lvert_{p\to p^{-1}} = (-1)^{3}p^{-3s+2}\zeta^{\leq}_{G,N,p}(s)$
IT=(3,9)	$\zeta(s)\zeta(s-1)^2\frac{1+2^{-(s-1)}+2^{-(2s-3)}}{\zeta_2(s)}$	2	$\zeta^{\leq}_{G,N,p}(s)\lvert_{p\to p^{-1}} = (-1)^{3}p^{-3s+2}\zeta^{\leq}_{G,N,p}(s)$
IT=(3,29)	$\zeta(s)\zeta(s-1)^2\frac{1+2^{1-s}}{\zeta_2(s)\zeta_2(s-1)}$	2	$\zeta^{\leq}_{G,N,p}(s)\lvert_{p\to p^{-1}} = (-1)^{3}p^{-3s+2}\zeta^{\leq}_{G,N,p}(s)$
IT=(3,33)	$\zeta(s)\zeta(s-1)^2\zeta_2(s)^{-1}\zeta_2(s-1)^{-2}$	2	$\zeta^{\leq}_{G,N,p}(s)\lvert_{p\to p^{-1}} = (-1)^{3}p^{-3s+2}\zeta^{\leq}_{G,N,p}(s)$

In the table and below, χ_3 and χ_4 denotes the primitive Dirichlet characters $\mod 3$ and $\mod 4$ respectively and $L(s,\chi)$ is the L-series associated to χ.

4.2. **The 3-dimensional AB-groups with non-abelian Fitting subgroup.** The 3-dimensional almost Bieberbach groups with non-abelian Fitting subgroup are presented with their names as in [D]. There are seven infinite families, and each family is parameterized with the natural numbers. If E is a member in any of these families, we always denote by N the Fitting subgroup of E. We present $\zeta^{\leq}_{E,N}(s)$ (depending on some parameter), its abscissa of convergence $\alpha^{\leq}(E,N)$ and a functional equation which is satisfied for all but a finite number of local factors of $\zeta^{\leq}_{E,N}(s)$. The parameters which appear in the expression of these zeta functions are explained in [S].

4.2.1. *The AB-groups of type $Q = p1$.*

$$\zeta^{\leq}_{E}(s) = \prod_{p\mid k} \frac{\zeta_p(s)\zeta_p(s-1) - p^{(2-s)(v_p(k)+1)}\zeta_p(2s-2)\zeta_p(2s-3)}{\zeta_p(s)\zeta_p(s-1) - p^{(2-s)}\zeta_p(2s-2)\zeta_p(2s-3)}$$

$$\cdot \frac{\zeta(s)\zeta(s-1)\zeta(2s-2)\zeta(2s-3)}{\zeta(3s-3)}, \quad \alpha^{\leq}(E) = 2,$$

$$\zeta^{\leq}_{E,p}(s)|_{p\to p^{-1}} = (-1)^3 p^{-3s+3}\zeta^{\leq}_{E,p}(s),\ p\nmid k.$$

4.2.2. *The AB-groups of the type $Q = p2$.*

$$\zeta^{\leq}_{E,N}(s) = \frac{\zeta(s-1)\zeta(s-2)\zeta(2s-1)\zeta(2s-2)}{\zeta(3s-3)} \cdot \frac{\zeta_2(3s-3)}{\zeta_2(2s-1)\zeta_2(2s-2)}$$

$$\cdot \prod_{p\neq 2, p\mid q} \frac{\zeta_p(s-1)\zeta_p(s-2) - p^{-s(v_p(q)+1)}\zeta_p(2s-1)\zeta_p(2s-2)}{\zeta_p(s-1)\zeta_p(s-2) - p^{-s}\zeta_p(2s-1)\zeta_p(2s-2)}$$

$$\alpha^{\leq}(E,N) = 3, \quad \zeta^{\leq}_{E,N,p}(s)|_{p\to p^{-1}} = (-1)^3 p^{-3s+3}\zeta^{\leq}_{E,N,p}(s), \quad p\nmid 2q.$$

4.2.3. *The AB-groups of type $Q = pg$.*

$$\zeta^{\leq}_{E,N}(s) = \zeta_2(s-2)\left(\zeta_2(s-1) - 2^{-(s-2)(v_2(q)+1)}\zeta_2(2s-3)\right)\cdot$$

$$\cdot \prod_{p\neq 2, p\mid q} \frac{\zeta_p(s)\zeta_p(s-1) - p^{(2-s)(v_p(q)+1)}\zeta_p(2s-2)\zeta_p(2s-3)}{\zeta_p(s)\zeta_p(s-1) - p^{2-s}\zeta_p(2s-2)\zeta_p(2s-3)}$$

$$\cdot \frac{\zeta(s)\zeta(s-1)\zeta(2s-2)\zeta(2s-3)}{\zeta(3s-3)}, \quad \alpha^{\leq}(E,N) = 2,$$

$$\zeta^{\leq}_{E,N,p}(s)|_{p\to p^{-1}} = (-1)^3 p^{-3s+3}\zeta^{\leq}_{E,N,p}(s),\ p\nmid 2q.$$

4.2.4. *The AB-groups of type $Q = p2gg$.*

$$\zeta^{\leq}_{E,N}(s) = \prod_{p\neq 2, p|q} \frac{\zeta_p(s-1)^2 - p^{-(s-1)(v_p(q)+1)}\zeta_p(2s-2)^2}{\zeta_p(s-1)^2 - p^{-(s-1)}\zeta_p(2s-2)^2} \cdot \frac{\zeta(s-1)^2\zeta(2s-2)^2}{\zeta(3s-3)}, \quad \alpha^{\leq}(E,N) = 2,$$

$$\zeta^{\leq}_{E,N,p}(s)|_{p\to p^{-1}} = (-1)^3 p^{-3s+3}\zeta^{\leq}_{E,N,p}(s), \; p \nmid 2q.$$

4.2.5. *The AB-groups of type $Q = p4$.*

$$\begin{aligned}\zeta_{E,N}(s) = \zeta_2(s-1)\prod_{p\neq 2}\Big[&\zeta_p(s)(\zeta_p(2s-2) - p^{-s(v_p(q)+1)}\zeta_p(4s-2)) \\ &+ (\chi_4(p)+1)\zeta_p(s)p^{-(s-1)}(\zeta_p(s-1)\zeta_p(2s-2) - p^{-s(v_p(q)+2)}\zeta_p(4s-2)\zeta_p(2s-1))\Big] \\ &\cdot \prod_{p|2q}\frac{L(3s-2,\chi_4,p)}{\zeta_p(s-1)\zeta_p(2s-1)L(s-1,\chi_4,p)L(2s-1,\chi_4,p)} \\ &\cdot \frac{\zeta(s-1)\zeta(2s-1)L(s-1,\chi_4)L(2s-1,\chi_4)}{L(3s-2,\chi_4)}, \quad \alpha^{\leq}(E,N) = 2,\end{aligned}$$

$$\zeta^{\leq}_{E,N,p}(s)|_{p\to p^{-1}} = (-1)^3\chi_4(p)p^{-3s+2}\zeta^{\leq}_{E,N}(s), \; p\nmid 2q.$$

4.2.6. *The Bieberbach groups of type $Q = p3$.*

$$\begin{aligned}\zeta^{\leq}_{E,N}(s) = \zeta_3(s-1)\prod_{p\neq 3, p|q}\zeta_p(s)\Big[&(\zeta_p(2s-2) - p^{-s(v_p(q)+1)}\zeta_p(4s-2)) + \\ &(1+\chi_3(p))\, p^{-(s-1)}(\zeta_p(s-1)\zeta_p(2s-2) - p^{-s(v_p(q)+2)}\zeta_p(4s-2)\zeta_p(2s-1))\Big] \\ &\cdot \prod_{p|q}\frac{L(3s-2,\chi_4,p)}{\zeta_p(s-1)\zeta_p(2s-1)L(s-1,\chi_3,p)L(2s-1,\chi_3,p)} \\ &\cdot \frac{\zeta(s-1)\zeta(2s-1)L(s-1,\chi_3)L(2s-1,\chi_3)}{L(3s-2,\chi_3)}, \quad \alpha^{\leq}(E,N) = 2,\end{aligned}$$

$$\zeta^{\leq}_{E,N,p}(s)|_{p\to p^{-1}} = (-1)^3 p^{-3s+2}\chi_4(p)\zeta^{\leq}_{E,N,p}(s), \; p\nmid 3q.$$

4.2.7. *The Bieberbach groups of type $Q = p6$.*

$$\zeta^{\leq}_{E,N}(s) = \zeta_2(2s-2)\zeta_3(s-1)\prod_{p\neq 2,3}\zeta_p(s)\Big[(\zeta_p(2s-2) - p^{-s(v_p(k)+1)}\zeta_p(4s-2))+$$

$$(1+\chi_3(p))p^{-(s-1)}\left(\zeta_p(s-1)\zeta_p(2s-2) - p^{-s(v_p(k)+2)}\zeta_p(4s-2)\zeta_p(2s-1)\right)\Big]$$

$$\prod_{p\neq 2,3; p|k}\frac{L(3s-2,\chi_4,p)}{\zeta_p(s-1)\zeta_p(2s-1)L(s-1,\chi_3,p)L(2s-1,\chi_3,p)}$$

$$\cdot\frac{\zeta(s-1)\zeta(2s-1)L(s-1,\chi_3)L(2s-1,\chi_3)}{L(3s-2,\chi_3)}, \quad \alpha^{\leq}(E,N) = 2,$$

$$\zeta^{\leq}_{E,N,p}(s)|_{p\to p^{-1}} = (-1)^3 p^{-3s+2}\chi_3(p)\zeta^{\leq}_{E,N,p}(s),\ p\nmid 6q.$$

Acknowledgements

The author's work was supported by CONICET and part of it was done during his visit to KULeuven-Kulak with an ERASMUS scholarship. The author thanks to Karel Dekimpe and Paulo Tirao for many suggestions that influenced the work. He is also very grateful with Christopher Voll for many stimulating conversations.

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